BΓ School

Homotopy Theory of Foliations

- 趣旨
- 日時・場所
- プログラム
- 題目とアブストラクト
- 演習
- 連絡先
- アクセス
- リンク
    "Seminar on Geometry, Dynamics and Foliations"
    (2013 年 9 月 21 日，23 日，25 日，於 中央大学)
    "Geometry and Foliations 2013"
    (2013 年 9 - 14 日，於 東京大学数理科学研究科)

- **趣旨**

  BGamma School は葉層構造におけるホモトピー論について、
  これまでの主たる結果 (h-原理等) を概観し、最近の進展まで紹介することを目的とした研究集会です。
  また大学院生向けに exercise class も企画しております。

- **日時・場所**

  2013 年 9 月 17 日 (火) – 9 月 19 日 (木)
  112-8551 東京都文京区春日 1-13-27 中央大学理工学部後楽園キャンパス 5 号館 1 階

- **プログラム** (tentative)

	17 (Tue)	18 (Wed)	19 (Thu.)
10:00-11:00		Vogt-2	Vogt-3
11:20-12:20		Asuke-1	Asuke-2
12:20-14:00		Lunch	Lunch
13:00-14:00		EC	EC
14:00-15:00	Vogt-1	Meigniez-2	Meigniez-3
15:20-16:20	Meigniez-1	Eynard-Bontemps	Schweitzer
16:40-17:40	Mitsumatsu	Vogel/Bowden	Morita
18:00-19:30	EC	EC	WP

EC : 演習（Exercise Class with tutors）
WP : ワインパーティー (18:00-21:00)

- **題目とアブストラクト**

  Elmar Vogt (Freie Universität Berlin):

  1. Why $B\Gamma$ and what is it?
  2. Simplicial constructions: other models for $B\Gamma$
  3. De Rham theory for $B\Gamma$ [abstract]

  Gaël Meigniez (Université de Bretagne-Sud):

  Thurston's h-principle [abstract]

  Yoshi Mitsumatsu (Chuo University):

  Thurston's h-principle in $(4,2)$-dimension: Applications and proof [abstract]

  Taro Asuke (University of Tokyo):

  Construction of secondary characteristic classes for foliations using the Chern-Weil theory [abstract]

  Hélène Eynard-Bontemps (Université Pierre et Marie Curie):

  Connected components of the space of smooth codimension one foliations on a closed 3-manifold [abstract]

  Thomas Vogel (Max Planck Inst. Bonn) / Jonathan Bowden (Augsburg University):

  Contact structures and foliations on 3-manifolds [abstract]

  Paul A. Schweitzer, S.J. (PUC-Rio) :

  Gelfand-Fuks cohomology and the cohomology of $B\Gamma_q$ [abstract]

  Shigeyuki Morita (University of Tokyo, Emeritus):

  A few problems on the classifying spaces of foliations [abstract]

- **Exercise class** (click here)

- **連絡先**

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- **アクセス**

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  都営地下鉄三田線・大江戸線春日駅下車，徒歩７分
  詳しくは こちら へ

- **Scientific Committee**

  Yoshi Mitsumatsu (Chuo University)

  Gaël Meigniez (Université de Bretagne-Sud)

  Elmar Vogt (Freie Universität Berlin)

- **Organizing Committee**

  Yoshi Mitsumatsu (Chuo University)

  Shigeaki Miyoshi (Chuo University)

  Noboru Ogawa (Chuo University)

アブストラクト

Lecture 1. Why $B\Gamma$ and what is it?
Lecture 2. Simplicial constructions: other models for $B\Gamma$
Lecture 3. De Rham theory for $B\Gamma$

Elmar Vogt

1. Why $B\Gamma$ and what is it?

We introduce Haefliger structures and indicate the rôle they play in answering certain classification problems for foliations. Finally after having given a second description of Haefliger structures we describe a model for the classifying space $B\Gamma_q$ of codimension q Haefliger structures together with the universal codimension q Haefliger structure on $B\Gamma_q$.

2. Simplicial Constructions: other Models for $B\Gamma$

Using different realizations of simplicial spaces we describe models for classifying spaces of topological groupoids or, more generally, of topological categories. We investigate some manipulations with these spaces which lead to an outline of a proof of the theorem of Graeme Segal (Classifying spaces related to foliations , Topology 17, 1978, 367-382) that $B\Gamma_q$ is weakly homotopy equivalent to the classifying space of the discrete monoid of smooth embeddings $\mathbb{R}^q \to \mathbb{R}^q$. If time permits we indicate how this approach might be used to give a proof of the Thurston-Mather theorem relating the discrete group of compactly supported diffeomorphisms of $\mathbb{R}^q$ with $B\Gamma_q$. Gaël Meigniez will give his recent proof of this theorem in the second lecture of this morning.

3. De Rham Theory for $B\Gamma$

The groupoid $B\Gamma_q$ of germs of local diffeomorphisms of $\mathbb{R}^q$ is a smooth but non-Hausdorff q-manifold. Thus the nerve of the corresponding category is a simplicial manifold, and thus using the de Rham complex on each of the manifolds of the simplicial manifold we obtain a double complex. The cohomology of the associated total complex should have some relationship with the real singular cohomology of the classifying space. If all manifolds are Hausdorff it actually computes the real singular cohomology. If not there is at least some map from the de Rham type cohomology to the singular cohomology, and the secondary characteristic classes are actually in this image. But going a little further using open covers of our non-Hausdorff manifolds by open Hausdorrf subsets leads if the covers are properly arranged to be compatible with the simplicial boundary maps to a calculation of the singular cohomology of the classifying space. Apart from the very last step all the above is contained in Bott, Shulman, Stasheff: On the de Rham Theory of Certain Classifying spaces, Advances of Math 20, 1976, 43-56. The final result is a special case of a theorem of Crainic and Moerdijk from 1998 but uses a simpler language.

Thurston's h-principle

Gaël Meigniez

These talks aim to report Thurston's proof (to this day, the only one) of his h-principle for foliations of codimension two and more. Then, depending on time, I'll explain the new proof of the Mather-Thurston homology equivalence by the same techniques.

Contact structures and foliations on 3-manifolds

Thomas Vogel / Jonathan Bowden

Eliashberg and Thurston have shown that all foliations (except foliations by spheres on $S^2 \times S^1$) can be approximated by contact structures provided that the foliation is sufficiently smooth and the underlying manifold is closed. We will explain the main ideas of this theorem and discuss the uniqueness of the contact structure approximating a foliation. This will then be applied to deduce topological properties of the space of taut foliations. (This talk is somewhat independent from the lectures about Haefliger's classifying spaces since we will discuss an equivalence relation on the space of taut foliations on a closed 3-manifold which is different from the equivalence relation relevant for $B\Gamma$.)

A few problems on the classifying spaces of foliations

Shigeyuki Morita

This is a continuation of my talk at the conference "Geometry and Foliations, 2013" given in the preceding week. Here we expand some part of it and focus on the difficult problem of determining homotopy types of various universal spaces which appear in the theory of foliations. The most important one is Haefliger's classifying space for Γ_1-structures in the smooth category.

Applications and a special proof of Thurston's h-principle for codimension $q > 1$ in the case of 2-dim foliations on 4-manifolds

Yoshihiko Mitsumatsu

This talk has the following purposes:

- By presenting some applications of Thurston's h-principle for foliations of codimension $q > 1$, we would like to draw attentions and to provide motivations for the homotopy theory of foliations.
- By explaining a special proof of the principle for the case of 2-dimensional foliations on 4-manifolds, which is due to Haefliger and Thurston, the talk serves as an introduction to especially Gaël Meigniez's minicourse and also presents an interesting application of the simplicity of some group of diffeomorphisms.

Section 1.

First we recall the statement of Thurston's h-principle and specialize it to the case of 2-dimensional foliations. It turns out to be that if we have a plane field on a manifold, then it is homotopically deformed as plane fields into an integrable one, i.e., a 2-dim foliation. This specialization stands on the $(q+1)$-connectedness of $B\overline{\Gamma}_q$ which is due to Haefliger (up to q-connectedness) and Mather-Thurston. Some parts of this connectivity might be presented as exercises which are common to the minicourses by Elmar Vogt and by Gaël Meigniez.

Section 2.

Then as an application in (4,2)-dim setting, we show that the existence of a foliation with prescribed compact leaf of a closed transversal is determined by cohomological informations.

Section 3.

As a further application of this, we can formulate and assure the existence of turbulization and other modifications. Also T^2-knot in $\mathbb{R}^4$ is always possible to realize as a compact leaf of a foliation on $\mathbb{R}^4$ which is trivial at infinity. For some special T^2-knot such as the trivial one or spun-knots we can give an explicit construction of foliations. The method originated in George Reeb's construction.

Section 4.

Almost the same construction is used in the special proof of the h-principle in (4,2)-dimensional case, which is the final topic in this talk. The simplicity and the simply-connectedness of $\mathit{Diff}_c(\mathbb{R}^2)$ play important roles. From three major steps of Thurston's proof for the general case, namely, jiggling lemma, civilization, and, inflation, we do not need the final one. For the first two, we should refer to Gaël Meigniez's minicourse.

References

[1] André Haefliger, Feuilletages sur les variétés ouvertes, *Topology* **9** (1970), 183–194.

[2] Y. Mitsumatsu and E. Vogt, Foliations and compact eaves on 4-manifolds I. Realization and self-intersection of compact leaves, in *Groups of Diffeomorphisms*, 415–442. Advanced Studies in Pure Mathematics **52**, Math. Soc. Japan, 2008.

[3] W. P. Thurston, Foliations and groups of diffeomorphisms, *Bull. Amer. Math. Soc.*, **80** (1974), 304–307.

[4] W. P. Thurston, The theory of foliations of codimension greater than 1, *Comment. Math. Helv.*, **49** (1974), 214–231.

Construction of secondary characteristic classes for foliations using the Chern-Weil theory

Taro Asuke

T.B.A

Connected components of the space of smooth codimension one foliations on a closed 3-manifold

Hélène Eynard-Bontemps

We are interested in the topology of the space $\mathcal{F}ol(M)$ of C^∞ transversely oriented codimension one foliations on a given closed oriented 3-manifold M. We make no difference between such a foliation and its tangent plane field, that is between foliations and *integrable plane fields*, and thus think of $\mathcal{F}ol(M)$ as a subset of the space $\mathcal{P}(M)$ of smooth plane fields on M, endowed with the usual C^∞ topology.

Most plane fields are not integrable: non integrable plane fields actually form a dense open subset of $\mathcal{P}(M)$. It is known since the late 60's, however, that any closed 3-manifold admits a smooth codimension one foliation. Moreover, according to works of J. Wood and W.P. Thurston, any smooth plane field can be deformed into a smooth foliation. In other words, the map $\pi_0\mathcal{F}ol(M) \xrightarrow{\iota_*} \pi_0\mathcal{P}(M)$ induced by the inclusion $\mathcal{F}ol(M) \overset{\iota}{\hookrightarrow} \mathcal{P}(M)$ is *surjective*. It is then tempting to ask whether this inclusion is actually a weak homotopy equivalence – or equivalently whether *foliations satisfy Gromov's h-principle* –, especially as such a result was obtained by Eliashberg in the closely related field of (overtwisted) contact structures. In this talk, we will present a first step in this direction:

Theorem *The inclusion of $\mathcal{F}ol(M)$ in $\mathcal{P}(M)$ induces a bijection between the sets of connected components of both spaces.*

Gelfand-Fuks cohomology and the cohomology of $B\Gamma_q$

Paul A. Schweitzer, S. J.

1. Definition of Gelfand-Fuks cohomology. The Gelfand-Fuks cohomology of a smooth manifold M^n is the cohomology of the Lie algebra $\mathcal{X}(M)$ of smooth vector fields on M with coefficients in $\mathbb{R}$ [2]. If $A^k(M)$ denotes the vector space of continuous alternating k-forms on $\mathcal{X}(M)$, then the differential $d^k : A^k(M) \to A^{k+1}(M)$ for $\alpha \in A^k(M)$ is defined as usual by setting

$$d\alpha(X_0, X_1, \ldots, X_k) = \sum_{0\leq i\leq j\leq k} (-1)^{i+j}\alpha([X_i, X_j], X_0, \ldots, \hat{X}_i \ldots, \hat{X}_j, \ldots, X_k)$$

where $X_1, \ldots, X_k \in \mathcal{X}(M)$. Then $d^{k+1} \circ d^k = 0$ and the Gelfand-Fuks cohomology of M in degree k is defined to be

$$H^k_{GF}(M) = \ker(d^k)/\mathrm{im}(d^{k-1}).$$

Gelfand and Fuks showed that for a compact manifold M the cohomology ring $H^*_{GF}(M)$ is finite-dimensional [2].

For $M = \mathbb{R}^n$, the Gelfand-Fuks cohomology $H^*_{GF}(\mathbb{R}^n)$ turns out to be the real cohomology of a finite CW-complex Y_n. If $p : E \to B = BU_n$ is the universal principal U_n bundle and X_n is the $2n$-skeleton of the base space B (which is well-defined since B is a CW-complex with only even-dimensional Schubert cells), then $Y_n = p^{-1}(X_n)$. At first it seemed that $H^*_{GF}(M)$ might be an essentially new invariant of smooth manifolds, but that turned out to be false. In fact, Bott's conjecture affirming that $H^*_{GF}(M)$ is isomorphic to the real cohomology of the space of sections of the fiber bundle with fiber Y_n associated to the tangent bundle TM was proven by several authors. (Note that $O_n \subset U_n$ acts on Y_n, so the bundle in question is well-defined.)

2. Calculation of $H^*_{GF}(\mathbb{R}^q)$. Using the Lie derivative with respect to the radial vector field $R = \sum_{i=1}^q \partial/\partial x_i$ on $\mathbb{R}^q$, one shows that the inclusion $A^q_{pt}\mathbb{R}^q \subset A^q\mathbb{R}^q$ of the alternating q-forms with support at the origin induces an isomorphism on homology. Now $A^q_{pt}\mathbb{R}^q$ is generated by the Dirac operators δ^i, which are defined $\delta^i(\sum_i f_i(x)\partial/\partial x_i = f_i(0)$, and their partial derivatives $\delta^i_j, \delta^i_{jk}$, etc. This permits explicit calculation of $H^*_{GF}(\mathbb{R}^q)$ (see Bott's notes [3] pp. 125-151).

Let $E(u_1, u_3, \ldots, u_{2k+1})$ be the exterior algebra on the elements of odd index up to $2k + 1$, the largest odd integer not greater than q. Let $\mathbb{R}[c_1, \ldots, c_q]$ be the polynomial algebra generated by the elements c_i with $\deg(c_i) = 2i$ and $du_i = c_i$, and denote the truncated polynomial algebra $\mathbb{R}[c_1, \ldots, c_q]/I_{2q}$, where I_{2q} is the ideal of elements of degree greater than $2q$, by $\widehat{\mathbb{R}}[c_1, \ldots, c_q]$. Then define

$$WO_n = E(u_1, u_3, \ldots, u_{2k+1}) \otimes \widehat{\mathbb{R}}[c_1, \ldots, c_q].$$

Theorem $H^k_{GF}(\mathbb{R}^q) \approx H^*(WO_q)$.

Corollary $H^k_{GF}(\mathbb{R}^q)$ *has nonzero elements in degrees* $k = 2q + 1$ *and* $k = q^2 + 2q$, *but vanishes for* $k < 2k + 1$ *or* $k > q^2 + 2q$.

The nonzero class $[u_1c_1^q] \in H^{2q+1}_{GF}(\mathbb{R}^q)$ corresponds to the Godbillon-Vey class.

3. Characteristic classes of foliations. The cohomology classes in $H^*_{GF}(\mathbb{R}^q)$ act as characteristic classes of a codimension q foliation $\mathcal{F}$ on a manifold M^n, since there is a homomorphism $\phi_{\mathcal{F}} : H^*_{GF}(\mathbb{R}^q) \to H^*(M;\mathbb{R})$. (See Bott-Haefliger [1].) Since the homomorphisms $\phi_{\mathcal{F}}$ are functorial, they induce a universal homomorphism

$$\phi : H^*_{GF}(\mathbb{R}^q) \to H^*(B\Gamma_q;\mathbb{R})$$

where $B\Gamma_q$ is the Haefliger classifying space for codimension q foliations. Bott and others conjectured that the homomorphism ϕ should be injective, so that the characteristic classes given by the Gelfand-Fuks cohomology would be independent. Partial results by Heitsch, Kamber and Tondeur, and Hurder, among others, support this conjecture, but the complete proof still seems to be very distant.

The cohomology of $B\Gamma_q$ is not countably generated, as shown by Thurston's examples giving continuous variation of the Godbillon-Vey class for foliations on S^3 and other characteristic classes which vary continuously. Perhaps there is a relationship with coefficient automorphisms of $\mathbb{R}$ as a $\mathbb{Q}$-vector space which also show that $H^*(B\Gamma_q;\mathbb{R})$ is immense.

To define $\phi_{\mathcal{F}}$ for a foliation $\mathcal{F}$ on M, consider the Lie algebra $a(\Gamma)$ of "formal Γ vector fields" as the inverse limit of the k-jets at 0 of smooth vector fields:

$$a(\Gamma) = \varprojlim a^k(\Gamma).$$

Here Γ is the pseudogroup of diffeomorphisms of open sets in $\mathbb{R}^q$. The cohomology of the continuous alternating forms on $a(\Gamma)$ coincides with that of $A^*_{pt}\mathbb{R}^q$. If the foliation $\mathcal{F}$ is defined locally by submersions $f_U : U \to \mathbb{R}^q$, the k-jets of f_U at points of U can be used to define $\phi_{\mathcal{F}}$. (See Bott-Haefliger [1] p. 1041.)

There are homomorphisms analogous to ϕ for the classifying spaces of foliations that are transversely holomorphic or transversely symplectic or have trivialized normal bundles. Partial results on independence of the corresponding characteristic classes have been obtained by Asuke in the first case and by various others in the second case.

References

[1] R. Bott and A. Haefliger, On the characteristic classes of Γ-foliations, Bull. Amer. Math. Soc. 78 (1972), 1039-1044.

[2] I.M. Gelfand and D.B. Fuks, Cohomologies of the Lie algebra of tangent vector fields of a smooth manifold, Functional Anal. Appl. 3 (1969), 194-210.

[3] M. Mostow and J. Perchik, Notes on Gel'fand Fuks cohomology and characteristic classes, in *Proceedings of the 11th Annual Holiday Symposium at New Mexico State Univ., Dec. 27-31, 1973*, Dept. of Mathematical Sciences, New Mexico State Univ., (1975), 1-220.

Exercise sets 1 and 2

For the lectures by Elmar Vogt, Gael Meigniez, and Yoshi Mitsumatsu

(provisional version)

$B\overline{\Gamma}_q$ is the homotopy fibre of the classifying map $B\Gamma_q \to BO_q$ of the normal bundle and homotopically classifies Γ_q-structures with trivialized normal bundles.

It is known that $B\overline{\Gamma}_q$ is $(q+1)$-connected, i.e., $\pi_i(B\overline{\Gamma}_q) = 0$ for $0 \leq i \leq q+1$. The q-connectedness is due to Haefliger. Up to $q-1$-connectedness it is shown using point foliations (i.e., codimension-q foliations on q-manifolds) and standing on Haefliger's theorem and the theory of Gromov-Phillips.

Exercise 1-1(easy) Show $\pi_0(B\overline{\Gamma}_1) = 0$ in the following sense. Show that a parallelized $S^0 \times \mathbb{R}^1$ (there are 4 such structures) is a parallelized submanifold of $\mathbb{R}^1$ with the standard parallelization.

Exercise 1-2 Show $\pi_1(B\overline{\Gamma}_2) = 1$ in the following sense. Show that with any parallelization $S^1 \times \mathbb{R}^1$ (there are $\mathbb{Z}$-many such structures) is immersed into $\mathbb{R}^2$ with the standard parallelization.

Now Haefliger's theorem implies any $\overline{\Gamma}_2$-structure on $S^1 \times \mathbb{R}^1$ is contractible and thus also $\pi_1(B\overline{\Gamma}_2) = 1$.

Exercise 1-3 By using 1-2, show the following result due to Claude Roger.

Any $\mathbb{R}^2$-bundle over S^2 can be the normal bundle to some $\overline{\Gamma}_2$-structure on S^2.

This is nothing but the surjectivity of the homomorphism $\pi_2(B\Gamma_2) \to \pi_2(BO_2)$ so that from the homotopy long exact sequence formally it follows immediately from 1-2.

Exercise 2-1 Show $\pi_1(B\overline{\Gamma}_1) = 1$ in the following way.

First confirm that any $\overline{\Gamma}_1$-structure on S^1 is realized as a smooth foliation on $S^1 \times \mathbb{R}^1$ which is transverse to $\{*\} \times \mathbb{R}^1$.

Next change the zero-section homotopically which draws a very wild sine-like curve and make shorter the $\mathbb{R}^1$-component so that the foliation becomes so simple.

Then embed the resultant foliation into a trivial foliation on $S^1 \times \mathbb{R}^1$ and extend it over D^2 as $\overline{\Gamma}_1$-structure.

Exercise 2-2 Toward $\pi_2(B\overline{\Gamma}_1) = 0$.
Assume such a singular foliation ($\overline{\Gamma}_1$-structure) on S^2 is given that it has two centers and away from these centers it can be regarded as an foliated interval bundle over S^1.

Show that this foliation is homologically trivial. Being combined with 2-1, this implies that the given ($\overline{\Gamma}_1$-structure) on S^2 extends to D^3.

The key ingredients are the perfectness and the contractibility of $\mathit{Diff}_c\mathbb{R}$.

References

[1] André Haefliger, Feuilletages sur les variétés ouvertes, *Topology* **9** (1970), 183–194.

[2] Anthony Phillips, Smooth maps transverse to a foliation, *Bull. Amer. Math. Soc.*, **76** (1970), 792–797.

[3] Mikhael Gromov, Partial differential relations. *Ergebnisse der Mathematik und ihrer Grenzgebiete* (3) [*Results in Mathematics and Related Areas* (3)], 9. Springer-Verlag, Berlin, 1986.