Why BGamma and What is it

BGamma – School 2018

Chuo University, Tokyo, March 12-14, 2018

1 Why BGamma and what is it

Of the several equivalent definitions for foliations the following is the most suitable for our purpose:

1.1 Definition. *A foliation $\mathcal{F}$ of codimension q and dimension $n-q$ on an n-manifold M is represented by an open cover $\mathcal{U}$ of M and a family $\{h_U : U \to \mathbb{R}^q \mid U \in \mathcal{U}\}$ of submersions such that locally h_U and h_V differ by a local diffeomorphism of $\mathbb{R}^q$.*

This means that for any $x \in U \cap V$ there is a neighborhood W of $h_U(x)$ in $\mathbb{R}^q$ and an embedding $h_{W,VU} : W \to \mathbb{R}^q$ such that on a neighborhood of x in $h_U^{-1}(W)$ the maps $h_{W,VU} \circ h_U$ and h_V agree.

Two such foliation representatives are said to represent the same foliation if their union is again a foliation representative.

The foliation is called a C^r-foliation if for some representative all maps h_U and $h_{W,VU}$ are C^r-maps.

The leaves of a foliation are the maximal connected subsets one obtains by forming at most countable unions of connected components of $h_U^{-1}(\{y\}), U \in \mathcal{U}, y \in \mathbb{R}^q$. This does not depend on the representative of the foliation.

If $\mathcal{F}$ is C^r, $r \geq 1$, the subbundle $\tau(\mathcal{F})$ of the tangent bundle $T(M)$ of M whose fibre over $x \in M$ is the tangent plane of the leaf through the point x we call the tangent bundle of $\mathcal{F}$, and by abuse of language also the tangent plane field of $\mathcal{F}$.

If $\mathcal{F}$ is a C^r-foliation, $r \geq 1$, on M the quotient bundle $\nu(\mathcal{F}) := T(M)/\tau(\mathcal{F})$ is called the normal bundle of $\mathcal{F}$.

Although the differentiability class plays a very important rôle for foliations, we will most of the time consider exclusively C^∞ foliations, and only refer to the differentiability class when it really matters.

Ideally, we would like to classify foliations of codimension q on M up to leaf preserving diffeomorphisms, i. e. diffeomorphisms of M which map leaves of the first foliation diffeomorphically onto the leaves of the second (Actually, as the term "leaf preserving diffeomorphism" implies, it suffices to demand that the diffeomorphism of M maps leaves to leaves). But this is currently completely out of reach. Just consider 1-dimensional foliations on 3-manifolds.

Here are three weaker equivalence relations that have been investigated.

1.2 Definition. *The foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ on M are said to be*

(1) homotopic, if $\mathcal{F}_0, \mathcal{F}_1$ extends to a family $\mathcal{F}_t, 0 \leq t \leq 1$, of foliations on M which varies continously in the sense that the associated tangent plane fields define a homotopy of plane fields;

(2) *concordant, if there exists a foliation $\mathcal{F}$ of codimension q on a neighbourhood of $M \times I$ in $M \times \mathbb{R}$ whose leaves are transverse to $M \times \{i\}, i = 0, 1$, and with $\mathcal{F}|_{M\times\{i\}} = \mathcal{F}_i, i = 0, 1$. The foliation $\mathcal{F}$ is then called a concordance between $\mathcal{F}_0$ and $\mathcal{F}_1$.*

(3) *integrably homotopic, if there exists a concordance $\mathcal{F}$ between $\mathcal{F}_0$ and $\mathcal{F}_1$ which is transverse to every $M \times \{t\}, t \in [0, 1]$.*

Clearly, integrably homotopic implies concordant and homotopic. Furthermore, after fixing a Riemannian metric of $M \times I$, we can consider the gradient flow of the projection $p : M \times I \to I$ tangent to $\mathcal{F}$ to see that integrably homotopic foliations on a closed manifold M are diffeomorphic by a diffeomorphism smoothly isotopic to the identity. Thus integrable homotopy is a very strong equivalence relation for foliations on closed manifolds.

The situation is quite different on open manifolds, i. e. manifolds without compact components. For example, it is a nice exercise to show that any foliation of codimension q on $\mathbb{R}^n$ is integrably homotopic to the standard foliation given by the product $\mathbb{R}^{n-q} \times \mathbb{R}^q$. The idea is to first deform the foliation so that in $[-1, 1]^{n-q} \times [-1, 1]^q$ the foliation is standard, and then push everything outside of $[-1, 1]^{n-q} \times [-1, 1]^q$ off to infinity. This result gives hope that there might be some classification of integrable homotopy classes on open manifolds M which depend only on the homotopy type of M.

In homotopy theory one usually associates the notion of classifying space with a homotopy functor. A homotopy functor is a set valued contravariant functor F defined on some nice subcategory $\mathcal{C}$ of the category of topological spaces such that $F(f) = F(g)$ if f and g are homotopic maps. A classifying space for F is a space BF (preferably in $\mathcal{C}$) together with an element $u_F \in F(BF)$, called a universal element, such that the map

$$\begin{array}{ccc} [X, BF] & \to & F(X) \\ [f] & \mapsto & F(f)(u_F) \end{array}$$

is bijective for all X belonging to $\mathcal{C}$. Here $[X, Y]$ denotes the set of homotopy classes of maps between X and Y, and $[f]$ denotes the homotopy class of f.

Usually, homotopy functors satisfy additional properties which guarantee that classifying spaces exist, but we will not be concerned with this, since the functors that we will be dealing with all satisfy these conditions. What matters for us is the problem that assigning to any manifold M the set of foliations of codimension q on M does not constitute a contravariant functor. To make it into a functor we would need to associate in a natural way to any continuous map $f : N \to M$ and any codimension q foliation $\mathcal{F}$ on M a codimension q foliation $f^*(\mathcal{F})$ on N. That this is a problem, even if we only consider smooth maps, becomes apparent, if the dimension of N is less than q.

Looking at our definition, if $\mathcal{F}$ is a foliation on M of codimension q and $f : N \to M$ a map from the manifold N to M there is a natural foliation on N induced by f if f is transverse to $\mathcal{F}$ in the sense of the following definition.

1.3 Definition. *Given a foliation $\mathcal{F}$ on M and a manifold N, a map $f : N \to M$ is called transverse to $\mathcal{F}$ if for one representative $\{h_U : U \to \mathbb{R}^q \mid U \in \mathcal{U}\}$ (and consequently for all representatives) of $\mathcal{F}$ all maps $h_U \circ f : f^{-1}(U) \to \mathbb{R}^q$ are submersions. If we are dealing with C^r-foliations we should also require that f is C^r.*

In this case the family $\{h_U \circ f : f^{-1}(U) \to \mathbb{R}^q \mid U \in \mathcal{U}\}$ represents a foliation on N denoted by $f^*(\mathcal{F})$.

If we are dealing with C^r-foliations, $r \geq 1$, notice that the C^r-map f is transverse to $\mathcal{F}$ if and only if the map

$$T(N) \stackrel{df}{\to} T(M) \to \nu(\mathcal{F})$$

is a bundle epimorphism.

This problem was very successfully resolved - especially when considering the later developments - by André Haefliger in his 1958 thesis [Hae1], where he introduced for a general topological groupoid Γ the so-called Γ-structures. The Γ-structures of the groupoid Γ_q^r of germs of local C^r-diffeomorphisms of $\mathbb{R}^q$, the groupoid relevant for the study of C^r-foliations of codimension q, are nowadays called *Haefliger strucures*.

Γ-structures are a natural generalization to topological groupoids of the concept of principal G-bundles, where G is a topological group. Recall that a topological category is a small category C where the set of objects C_0 carries a topology and so does the set C_1 of all morphisms. Furthermore one requires that all structure maps are continuous. These are the maps $Id: C_0 \to C_1, Id(X) = id_X$, source $s: C_1 \to C_0$ and target $t: C_1 \to C_0$ and the composition $c: C_1 \times_s C_1 \to C_1$. Here $C_1 \times_s C_1 = \{(f,g) \in C_1 \times C_1 \mid s(f) = t(g)\}$ and $c(f,g) = f \circ g$.

A *topological groupoid* is then simply a topological category C where all morphisms are isomorphisms, and the map taking morphisms to their inverses is continuous. In this set-up a topological group is a topological groupoid where C_0 consists of a single object. For a topological groupoid C the space C_0 is called the *space of units*. Notice that Id is an embedding, so that we can identify C_0 with its image in C_1. This allows us to identify the category C with its morphism space C_1 and with the units a subspace of C, which we continue to call C_0.

In the bundle chart description, a principal G-bundle over the space X is given by an open cover $\mathcal{U} = (U_i)_{i \in J}$ of X and continuous maps $h_{ji}: U_i \cap U_j \to G, i, j \in J$, satisfying the *cocycle condition* $h_{kj}(x) \cdot h_{ji}(x) = h_{ki}(x)$ for all $x \in U_k \cap U_j \cap U_i$ and for all $i, j, k \in J$. We call such a description a cocycle. Two cocycles $(U_i, h_{ji})_{i,j \in J}$ and $(V_a, k_{ba})_{a,b \in A}$ are considered to be equivalent, if their union can be extended to another cocycle, i. e., if for $a \in A$ and $i \in J$ there exist continuous $l_{ai}, l_{ia}: U_i \cap V_a \to G$ such that the union of the covers together with the $h_{ji}, k_{ba}, l_{ai}, l_{ia}$ is again a cocycle.

The G-bundle associated to the cocycle $(U_i, h_{ji})_{i,j \in J}$ has as total space the quotient of the disjoint union

$$\bigsqcup_{i \in J} U_i \times G$$

by the equivalence relation

$$(x,g) \in U_i \times G \sim (x',g') \in U_j \times G, \text{ if and only if } x = x' \text{ and } g' = h_{ji}(x) \cdot g.$$

The cocycle condition guarantees that this is in fact an equivalence relation. It also implies that $h_{ii}(x) = 1$ for $x \in U_i$, and $h_{ij}(x) = (h_{ji}(x))^{-1}$ for $x \in U_i \cap U_j$.

The bundle map projects the equivalence class of $(x,g) \in U_i \times G$ onto x, and the right action of G on the total space is given by right multiplication on the second factor. Clearly, equivalent cocycles define G-bundles which are isomorphic. The isomorphism is given by mapping the class of $(x,g) \in U_i \times G$ to the class of $(x, l_{ai}(x)g) \in V_a \times G$, if $x \in V_a$. Again, this is well defined because of the cocycle condition.

The set of isomorphism classes of G-bundles over X is sometimes denoted by $H^1(X; \underline{G})$ because it can be interpreted as the first Čech cohomology group with coefficients in the sheaf $\underline{G}$ of germs of continuous maps from open sets of X to G.

All the above can be generalized to principal Γ-bundles where Γ is a topological groupoid, if one takes care of the fact that multiplication of morphisms is not always possible.

A right Γ-action on a space Y with respect to a continuous map $\sigma : Y \to \Gamma_0$ is a continuous map $Y \times_\sigma \Gamma \to Y, (y,\gamma) \mapsto y\gamma$, satisfying $\sigma(y\gamma) = s(\gamma)$ and $y(\gamma_1\gamma_2) = (y\gamma_1)\gamma_2$, whenever one of the two sides is defined. As before $Y \times_\sigma \Gamma = \{(y,\gamma) \mid \sigma(y) = t(\gamma)\}$.

1.4 Definition. *Let Γ be a topological groupoid. A principal Γ-bundle over the space X consists of the following data:*

(i) A space E, called the total space, together with continuous maps $p : E \to X$, called the projection map, and $u : E \to \Gamma_0$, called the unit map, and a right Γ-action on E with respect to u.

The data are required to satisfy the following conditions

(ii) $p(e\gamma) = p(e)$, whenever the left side is defined, and for all $e, e' \in E$ with $p(e) = p(e')$ there exists a unique $\gamma \in \Gamma$ with $e' = e\gamma$.

(iii) (Local triviality) For every $x \in X$ there exists a section $\sigma_U : U \to E$ of p, with U an open neighborhood of x in X, and a Γ-invariant homeomorphism $\varphi_U : p^{-1}(U) \to U \times_{u\circ\sigma_U} \Gamma$ such that $p \circ \varphi^{-1}(y,\gamma) = y$ for $y \in U$. Here Γ operates on $U \times_{u\circ\sigma_U} \Gamma$ by right multiplication on the second factor.

The difference between principal G-bundles, G a group, and principal Γ-bundles, Γ a groupoid, disappear completely when looking at bundle chart descriptions: simply replace G by Γ. The cocycle condition means that for $x \in U_i \cap U_j \cap U_k$ the product $h_{kj}(x)h_{ji}(x)$ is defined and equals $h_{ki}(x)$. The total space of the principal Γ-bundle associated to the cocycle (U_i, h_{ji}) is the quotient of

$$\bigsqcup_{i\in J} U_i \times_{h_{ii}} \Gamma$$

by the equivalence relation

$$(x,\gamma) \in U_i \times_{h_{ii}} \Gamma \sim (x',\gamma') \in U_j \times_{h_{jj}} \Gamma \text{ if and only if } x = x' \text{ and } \gamma' = h_{ji}(x)\gamma.$$

The equivalence of cocycles is defined as before, and equivalent cocycles define isomorphic principal Γ-bundles. The set of isomorphism classes of principal Γ-bundles is denoted by $H^1(X; \underline{\Gamma})$.

For historical reasons we call an equivalence class of Γ-cocycles, i. e. an element of $H^1(X; \underline{\Gamma})$, also a Γ-*structure.*

Clearly, Γ-cocycles and Γ-structures are functorial. If (U_i, h_{ji}) is a Γ-cocycle on X and $f : Y \to X$ is continuous, then $(f^{-1}U_i, h_{ji} \circ f)$ is a Γ-cocycle on Y, called the pullback of (U_i, h_{ji}) by f, and equivalent cocycles pull back to equivalent cocycles.

Before we move on to classifying spaces we establish the connection between foliations and Γ-structures. We start with a foliation atlas $(U_i, h_i : U_i \to U_i')_{i\in J}$. We had noticed earlier that the foliation on U_i is already determined by the submersion $p^2 \circ h_i : U_i \to \mathbb{R}^q$ where $p^2 : \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ is the projection onto the second factor. The special form of coordinate changes for foliation atlases translates to the statement, that for every $z \in U_i \cap U_j$ there is a neighborhood W of $p^2 \circ h_i(z)$ in $\mathbb{R}^q$ and a diffeomorphism h_{ji}^2 from W onto a subset of $\mathbb{R}^q$ such that $p^2 \circ h_j = h_{ji}^2 \circ p^2 \circ h_i$ on some neighborhood of z in $U_i \cap U_j$.

In fact, the last equation determines h^2_{ji} uniquely since $p^2 \circ h_i$ and $p^2 \circ h_j$ are submersions. This also implies that for $z \in U_i \cap U_j \cap U_k$ the local diffeomorphisms h^2_{ki} and $h^2_{kj} \circ h^2_{ji}$ of $\mathbb{R}^q$ agree on some neighborhood of $p^2 \circ h_i(z)$.

In our previous language this means that a foliation gives rise to a Haefliger structure, i. e. of an element of $H^1(M; \underline{\Gamma}_q)$, where Γ_q is the set of germs of local diffeomorphisms of $\mathbb{R}^q$ with the sheaf topology.

Recall that germs at $x \in N$ of smooth maps from the manifold N to another manifold Q is an equivalence class of smooth maps into Q defined in some neighborhood of x in N with two such maps being equivalent if they agree on some neighborhood of x. For each germ γ at $x \in X$ consider the sets $\{j_y f \mid y \in W\}$ with $f : W \to Q$ a representative of γ, and where $j_y f$ denotes the germ at y represented by f. Then these sets form a neighborhood basis for γ of a unique topology on the set $\Gamma(N, Q)$ of all germs of smooth maps from N to Q. This topology makes $\Gamma(N, Q)$ into a (highly non-Hausdorff) manifold of the same dimension as N and is called the sheaf topology. Notice that a continous map $Z \to \Gamma(N, Q)$ is near some point $z_0 \in Z$ of the form $z \mapsto j_{h(z)} f$ where h is continuous and defined near z_0 and f is some smooth function into Q defined on some neighborhood of $h(z_0)$.

Note that the space of units of Γ_q is naturally homeomorphic to $\mathbb{R}^q$.

Putting all this together we obtain:

1.5 Proposition. *A foliation of codimension q on the manifold M is a Haefliger structure of codimension q, i e. a Γ_q-structure, on M which has a representing cocycle (U_i, h_{ji}) such that all $h_{ii} : U_i \to \mathbb{R}^q$ are submersions. If one representing cocycle satisfies this condition then all do.*

Notice that the Γ_q-structure induced by a smooth map $f : N \to M$ from a foliation of codimension q on M represented by (U_i, h_{ji}) is a foliation if and only if the maps $h_{ii} \circ f : f^{-1}(U_i) \to \mathbb{R}^q$ are submersions for all i.

All these notions extend naturally to C^r foliations, $0 \le r \le \infty$, by considering the groupoid Γ^r_q of germs of local C^r-diffeomorphisms of $\mathbb{R}^q$.

On the category of paracompact spaces the functor $H^1(-, G)$ (for simplicity, we will write $H^1(-, G)$ instead of $H^1(-, \underline{G})$ from now on) which associates to a space X the set of isomorphism classes of G-principal bundles, G a topological group, is a homotopy functor, and there is the well-known infinite join construction $EG \to BG$ of John Milnor of a universal principal G-bundle over a classifying space BG for this functor.

There is a completely analogous construction $E\Gamma \to B\Gamma$ of a principal Γ-bundle for any topological groupoid Γ, which we will explain below. But it cannot classify Γ-structures since $H^1(-, \Gamma)$ is not a homotopy functor, not even on very simple spaces.

Here is the standard example. Consider the Γ_q-structure on $\mathbb{R}^q$ given by the cocycle $(\mathbb{R}^q, id_{\mathbb{R}^q})$. This actually is the foliation by points of $\mathbb{R}^q$. Let $f : \mathbb{R}^q \to \mathbb{R}^q$, be the constant map with image 0. Then f and $id_{\mathbb{R}^q}$ are homotopic, but the induced Γ_q-structures represented by $(\mathbb{R}^q, f)$ and $(\mathbb{R}^q, id_{\mathbb{R}^q})$ cannot be equivalent since the first one is not a foliation of codimension q.

So one forces a homotopy functor by simply making the following

1.6 Definition. *Let Γ be a topological groupoid. Then two Γ-structures z_0 and z_1 on the space X are called concordant if there exists a Γ-structure z on $X \times I$ such that the induced Γ-structure on $X \times \{i\}$ is z_i for $i = 0, 1$.*

Concordance is an equivalence relation, and we denote the set of concordance classes of Γ-structures on X by $\Gamma(X)$. Clearly $\Gamma(-)$ is a homotopy functor, and it does admit a classifying space.

There exist several constructions for a classifying space of Γ-bundles for topological groupoids Γ, most prominently the above mentioned Milnor construction as exposed by Haefliger in [Hae4], section 5. For us also the so-called thick realization $\|\Gamma\|$ of the associated simplicial space [Se2] will play a rôle.

The constructions for the classifying space work for any topological category C. For this, we follow [Se1]. Another question is, what these spaces actually classify. Even for a given groupoid, there are interesting geometrically different interpretations, which also will be of interest to us. For general topological categories C, we refer to [Se3], §4.

To any topological category C one associates the *simplicial space* $C_\bullet$. This is a sequence of spaces C_n, $n = 0, 1, \ldots$, and for each $n > 0$ maps $d_i : C_n \to C_{n-1}$, $0 \leq i \leq n$, and $s_i : C_{n-1} \to C_n$, $0 \leq i \leq n-1$ which are defined as follows.

C_0 and C_1 are as before the space of objects and morphisms.

For $n \geq 1$ the space C_n is the subspace of C_1^n consisting of n-tuples of composable morphisms.

$s_i : C_n \to C_{n+1}$ is defined by

$$s_i(f_1, \ldots, f_n) = \begin{cases} (t(f_1), f_1, \ldots, f_n) & i = 0 \\ (f_1, \ldots, f_i, s(f_i), f_{i+1}, \ldots, f_n) & 1 \leq i \leq n \end{cases}$$

$d_i : C_n \to C_{n-1}$ is defined for $n > 1$ by

$$d_i(f_1, \ldots, f_n) = \begin{cases} (f_2, \ldots, f_n) & i = 0 \\ (f_1, \ldots, f_i \circ f_{i+1}, \ldots, f_n) & 0 < i < n \\ (f_1, \ldots, f_{n-1}) & i = n \end{cases}$$

and $d_0(f) = s(f)$, $d_1(f) = t(f)$.

The maps d_i are called the i-th face maps, and the s_i are called the i-th degeneracy maps. Please note that there should in fact be a second index n for the d_i, s_j. But, and this is common usage, we will suppress it from our notation.

More generally, a *simplicial space* is a contravariant functor $X_\bullet$ from the category $\mathbb{N}$ to the category of (compactly generated) topological spaces $\mathcal{T}$. The objects of $\mathbb{N}$ are the ordered sets $\underline{n} = \{0, \ldots, n\}$, $n = 0, 1, \ldots$, and the morphisms are the weakly monotonous maps. We denote by X_n the image of $\underline{n}$. The strictly monotonous map $\delta_i : \underline{n-1} \to \underline{n}$ the image of which misses i then maps to a continuous map $d_i : X_n \to X_{n-1}$, called as above the i-th face map. Similarly, the surjective monotonous map $\sigma_i : \underline{n} \to \underline{n-1}$ which maps i and $i+1$ to i will be mapped to the i-th degeneracy map $s_i : X_{n-1} \to X_n$, , $i = 0, \ldots, n-1$. Every morphism of $\mathbb{N}$ can be written as a composition of the δ_i and σ_j, in fact there is a unique factorization of the form $\delta_{i_k} \circ \cdots \circ \delta_{i_0} \circ \sigma_{j_0} \circ \cdots \circ \sigma_{j_l}$ with $i_0 < \cdots < i_k$ and $j_0 < \cdots < j_l$.

So, for a simplicial space $X_\bullet$, it suffices to describe the face- and degeneracy maps, and they can be arbitrary, as long as they satisfy the dual of the relations satisfied by the δ_i and σ_j in $\mathbb{N}$.

If all X_n are discrete, we often talk of a *simplicial set*.

The *thick realization* $\|C\|$ *of a simplicial space* $C_\bullet$ is a quotient of

$$\coprod_{n\geq 0} C_n \times \Delta^n$$

where Δ^n is the standard n-simplex. The identifications are given by

$$(x_n, (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})) \sim (d_i x_n, (t_0, \ldots, t_{n-1}))\,,$$
$$x_n \in C_n\,,\ (t_0, \ldots, t_{n-1}) \in \Delta^{n-1}\,,\ 0 \leq i \leq n\,.$$

There is also the *thin realization* $|X|$ *of* $X_\bullet$. It is obtained from the thick realization by making the following additional identifications:

$$(x_{n-1}, (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_n)) \sim (s_i x_{n-1}, (t_0, \ldots, t_n))\,,$$
$$x_{n-1} \in C_{n-1}\,,\ (t_0, \ldots, t_n) \in \Delta^n\,,\ 0 \leq i \leq n-1\,.$$

What we call the Milnor construction is actually a slight variant of the original construction of Milnor. It does not make use of inverses (of morphisms) and thus can be applied to any topological category C (see e.g. [St]). The corresponding space will be called BC. As with the thick and thin realization one first passes to the associated simplicial space $C_\bullet$. Then one obtains BC from

$$\coprod_{n\geq 0}\ \coprod_{\sigma\in\Sigma_n} C_n \times \Delta^n_\sigma$$

by making the obvious face identifications. Here Σ_n is the set of n-faces Δ^n_σ of the standard infinite dimensional simplex Δ^∞ in $\mathbb{R}^\infty$.

There are obvious maps $BC \to \|C\| \to |C|$. The first one is by [tD1] always a homotopy equivalence, while the second one is a homotopy equivalence, if for all $n \geq 1$ the inclusion of the degenerate n-simplices into C_n is a cofibration ([tD1], Proposition 1. See also [Se2], Appendix A, Proposition A.1.(iv), if one assumes that the inclusion is a closed cofibration). A degenerate n-simplex is a point in the image of one of the $s_i : C_{n-1} \to C_n$.

1.7 Example. To get a first impression of the three realizations, consider the simplest case of the trivial category $C := *$ with one object and one morphism, and look at the n-skeleton $|C|_n$, $\|C\|_n$, and B_nC of the various realizations, i. e. the image of $C_n \times \Delta^n$ resp. the image of $\coprod_{\sigma\in\Sigma_n} C_n \times \Delta^n_\sigma$ in the Milnor case. Then $|C|_n$ is always a point, $\|C\|_n$ is contractible for even n and homotopy equivalent to the n-sphere for odd n, while B_nC is the n-skeleton of the infinite dimensional simplex Δ^∞.

Now let Γ be any topological groupoid. To make $B\Gamma$ (or$\|\Gamma\|$) into a full-fledged classifying space we need to equip it with a principal Γ-bundle, or equivalently with a Γ-cocycle. For this consider for each $k = 0, 1, \ldots$ the map $\tau_k : B\Gamma \to I = [0, 1]$ which maps the class of $(\gamma_1, \ldots, \gamma_n, (t_0, t_1, \ldots)) \in \Gamma_n \times \Delta^n_\sigma$ to t_k. This map is well defined and continuous. For $k < l$ we define a map $\gamma_{kl} : \tau_l^{-1}(0, 1] \cap \tau_k^{-1}(0, 1] \to \Gamma$ as follows. Pick any representative $(\gamma_1, \ldots, \gamma_n, t) \in \Gamma_n \times \Delta^n_\sigma$ of a point in this intersection. Let $i_0 < \cdots < i_n$ be the vertices of σ, and let $k = i_s$ and $l = i_t$. Then this point will be mapped to the composition $\gamma_{s+1}\gamma_{s+2}\cdots\gamma_t$. For $k > l$ we set $\gamma_{kl} = \gamma_{lk}^{-1}$, and $\gamma_{kk} : \tau_k^{-1}(0, 1] \to \Gamma$ maps the class of $(\gamma_1, \ldots, \gamma_n, t)$ to $t(\gamma_{s+1})$ if $k < i_n$ and otherwise to $s(\gamma_s)$. One checks that all these maps are is well defined, continuous, and satisfy the cocycle condition.

The corresponding Γ-structure u_Γ is universal for principal Γ-bundles over paracompact spaces.

Remember that any open covering of a paracompact space M has a countable locally finite refinement $(U_i)_{i\in\mathbb{N}}$ with a subordinate partition of unity $(t_i)_{i\in\mathbb{N}}$. Thus any Γ-structure on M has a representing cocycle of the form $(U_i,\gamma_{ji})_{i,j\in\mathbb{N}}$. A classifying map $f : M \to B\Gamma$ for the corresponding Γ-structure can be then obtained as follows. For $x \in M$ let $\{j_0 < j_1 < \cdots < j_k\} = \{j \in \mathbb{N} : t_j(x) > 0\}$ and let σ_x be the face of Δ^∞ spanned by $e_{j_0},\dots,e_{j_k}$, where $e_0, e_1,\dots$ is the standard basis of $\mathbb{R}^\infty$. Then $f(x)$ is the equivalence class of

$$((\gamma_{j_0j_1}(x),\dots,\gamma_{j_{k-1}j_k}(x)),\sum_{i=0}^{k} t_{j_i}(x)e_{j_i}) \in \Gamma_k \times \Delta^k_{\sigma_x} \quad \text{in } B\Gamma\,.$$

From this the first part of the following proposition due to Buffet–Lor [BuL] and Haefliger [Hae4] follows easily. The second part requires some manipulations concerning the Δ^∞-coordinates of $B\Gamma$.

1.8 Proposition.

(i) For each Γ-structure α on a paracompact space M there exists a continuous map $f : M \to B\Gamma$ such that $\alpha = f^*(u_\Gamma)$

(ii) If $f_0, f_1 : M \to B\Gamma$ are continuous maps such that $f_0^*(u_\Gamma) = f_1^*(u_\Gamma)$ then there is a homotopy h_t from f_0 to f_1 such that $f_t^*(u_\Gamma) = f_0^*(u_\Gamma)$ for all t.

The claim that $[M,B\Gamma] \to \Gamma(M)$,$[f] \mapsto f^*(u_\Gamma)$, is bijective for paracompact M follows easily from the proposition.

We can obtain a mildly more general point of view by introducing for any open cover $\mathcal{U} = (U_i)_{i\in J}$ of M its associated topological groupoid $\Gamma_\mathcal{U}$. Its space of objects (units) is $\coprod_{i\in J} U_i$, so objects are pairs (x,U_i) with $x \in U_i$. There is a morphism $(x,U_i) \to (y,U_j)$ if and only if $x = y$, and then there exists exactly one. So the set of morphisms is $\coprod_{(i,j)\in J\times J} U_i \cap U_j$, and we topologize it accordingly. A Γ-cocycle (U_i,γ_{ji}) is then simply a continuous functor $F : \Gamma_\mathcal{U} \to \Gamma$. The functor F induces a continuous map $BF : B\Gamma_\mathcal{U} \to B\Gamma$ or $\|F\| : \|\Gamma_\mathcal{U}\| \to \|\Gamma\|$ of classifying spaces, and also a continuous map $|F| : |\Gamma_\mathcal{U}| \to |\Gamma|$ for the thin realizations. So these maps for "classifying" the corresponding Γ-structure are always available.

To get a classifying map from M one needs to relate the spaces $B\Gamma_\mathcal{U}$ and $\|\Gamma_\mathcal{U}\|$ to M. The maps $B\Gamma_\mathcal{U} \to \|\Gamma_\mathcal{U}\| \to |\Gamma_\mathcal{U}|$ are homotopy equivalences. The first one is a homotopy equivalence for any topological category, and the the second one, because the union of degenerate n-simplices of $(\Gamma_\mathcal{U})_n$ is a topological summand of $(\Gamma_\mathcal{U})_n$ and therefore a closed cofibration.

There are obvious projections from these spaces to M, and if M is paracompact (and $J = \mathbb{N}$) we can with the help of partitions of unity produce sections for these projections. For the thin realization it is easy to prove that the section and projection are homotopy inverses, and therefore also the projections from $B\Gamma_\mathcal{U}$ and $\|\Gamma_\mathcal{U}\|$ are homotopy equivalences.

After this excursion about classifying space constructions we return to foliations.

The key result which allows to relate classification results of Γ^r_q-structures, i. e. C^r-Haefliger structures of codimension q, to classification results of C^r-foliations of codimension q is again due to A. Haefliger [Hae1]. He associated to every Haefliger structure γ of codimension q on M a foliated microbundle $(p_\gamma, i_\gamma, \mathcal{F}_\gamma)$ of rank q over M, and showed that the map $\gamma \mapsto (p_\gamma, i_\gamma, \mathcal{F}_\gamma)$ is a bijection. We recall the definition of a foliated microbundle. We limit ourselves to the smooth case, the changes to treat the C^r-case should be clear.

1.9 Definition. *A representative of a foliated microbundle of rank q over the n-manifold M consists of*

(i) a smooth submersion $p : E \to M$, where E is an $(n+q)$-manifold;

(ii) a continuous section $i : M \to E$ of p;

(iii) a smooth codimension q foliation $\mathcal{F}$ on E transverse to the fibers of p.

Two representatives $(E, p, i, \mathcal{F})$ and $(E', p', i', \mathcal{F}')$ are called equivalent, if there is an open neighborhood U of $i(M)$ in E and a smooth embedding $f : U \to E'$ such that $f \circ i = i'$ on M, $p' \circ f = p$ on U, and f maps leaves of the restriction of $\mathcal{F}$ to U to leaves of $\mathcal{F}'$.

There is a generalization of this concept to spaces M which are not necessarily manifolds. It will be useful in our context. But we will only indicate the necessary changes. Every point x of E is required to have neighborhood homeomorphic to $V \times \mathbb{R}^q$, with V some open neighborhood of $p(x)$ in M, and on this neighborhood p corresponds to the projection onto the first factor. Coordinate changes of these neighborhoods are supposed to be locally of the form $(y, t) \mapsto (y, k(t))$ with k being a smooth diffeomorphism between open sets of $\mathbb{R}^q$. So the slices $V \times \{t\}$, $t \in \mathbb{R}^q$ piece together to form leaves of a transversely C^r-foliation of E transverse to the fibers of p, with leaves which are locally homeomorphic to M.

Now let γ be a Γ_q-structure on M, represented by a cocycle $(U_i, \gamma_{ji})_{i,j \in J}$. The space of units of Γ_q is $\mathbb{R}^q$. So for every $i \in J$ we have the map $\gamma_{ii} : U_i \to \mathbb{R}^q$. We foliate each $U_i \times \mathbb{R}^q$ horizontally by the sets $U_i \times \{t\}$, $t \in \mathbb{R}^q$. The idea is now to piece the various $U_i \times \mathbb{R}^q$ near the graphs of the γ_{ii} together by using locally representatives of the $\gamma_{ji}(\gamma_{ii}(x))$, $x \in U_i \cap U_j$, for making the vertical adjustments.

In a little more detail, we make the following:

1.10 Claim. If M is paracompact then γ can be represented by a cocycle $(U_i, \gamma_{ji})_{i,j \in J}$ with the following property. For every $i \in J$ there exists a neighborhood O_i of the graph of γ_{ii} in $U_i \times \mathbb{R}^q$, and for each $(i, j) \in J \times J$ a homeomorphism $h_{ji} : O_i \cap (U_i \cap U_j \times \mathbb{R}_q) \to O_j \cap (U_i \cap U_j \times \mathbb{R}_q)$ such that

(i) h_{ji} preserves the first coordinate and

(ii) the germ at $\gamma_{ii}(x)$ of the restriction of h_{ji} to $O_i \cap (\{x\} \times \mathbb{R}^q)$ is $\gamma_{ji}(x)$.

The proof of the claim is an exercise in point set and sheaf topology. The space E_γ is then the union of the O_i glued together by the h_{ji}. The map $p_\gamma : E_\gamma \to M$ is on each O_i projection onto the first factor, the section i_γ maps $x \in U_i$ to $(x, \gamma_{ii}(x)) \in O_i$ and the transverse foliation $\mathcal{F}_\gamma$ is given on each $O_i \subset U_i \times \mathbb{R}^q$ by projection onto the second factor.

Also, if the Γ_q-structure associated to the transverse foliation is denoted by ε, then $i_\gamma^* \varepsilon = \gamma$. This allows us to pass back and forth between Γ_q-structures and foliated microbundles.

In addition, if M is a smooth manifold, then E_γ, p_γ, and $\mathcal{F}_\gamma$ are also smooth. The Γ-structure γ is a Γ-structure of a foliation $\mathcal{F}$ if and only if all maps h_{ii} are submersions. Then i_γ is smooth, the foliation $\mathcal{F}_\gamma$ is transverse to i_γ, and the connected components of the intersections of the leaves of $\mathcal{F}_\gamma$ with $i_\gamma(M)$ are the leaves of a foliation which projects diffeomorphically to $\mathcal{F}$.

As one further step, we look at the *normal bundle of a* Γ_q*-structure* γ. There is a natural continuous functor $\Gamma_q \to Gl(\mathbb{R}, q)$ which maps the germ of the local diffeomorphism f of $\mathbb{R}^q$ at the point $x \in \mathbb{R}^q$ to the derivative *df* of f at the point x. This maps any Γ_q-cocycle to a $Gl(\mathbb{R}, q)$-cocycle, and thus associates to any Γ_q-structure γ a q-dimensional real vector bundle: the normal bundle ν_γ of γ. If γ is actually the Γ_q-structure of a foliation $\mathcal{F}$ on the manifold M, then ν_γ is the normal bundle of the foliation $\mathcal{F}$.

If γ is a Γ_q-structure on a manifold M we may homotope i_γ to a smooth section of p_γ, which we call again i_γ. Then $i_\gamma^*\varepsilon_\gamma$may not be any longer equal to γ, but it is still concordant to γ. Also we may identify E_γ with a neighborhood of the 0-section of ν_γ and i_γ with this 0-section.

Given a manifold M of dimension n. The first obstruction one encounters, if one wants to put a foliation of codimension q on M is the existence of a q-field on M that can serve as the normal bundle ν to a foliation. If we have such a field, a foliation of codimension q transverse to this field is a Haefliger structure whose foliation, considered as a foliation on ν, is transverse to the 0-section of ν. So at least there must be some Γ_q-structure γ on M whose normal bundle has an imbedding into the tangent bundle $T(M)$ of M.

In fact, by two fundamental theorems of W. Thurston [Th2], [Th3], this is all you need to build a foliation on M which, as a Γ_q-structure , is concordant to γ.

In more detail, call two bundle monomorphisms $j_0, j_1 : \nu \to T(M)$ over M concordant if there is a bundle monomorphism $j : \nu \times I \to T(M \times I)$ over $M \times I$ with j restricted to $\nu \times \{i\}$ equal to j_i for $i = 0, 1$. Then we have

1.11 Theorem. *Let* γ *be a* Γ_q*-structure on the manifold* M *and* $j : \nu_\gamma \to T(M)$ *a bundle monomorphism. Then there exists a foliation* $\mathcal{F}$ *concordant to* γ *with* $\nu_{\mathcal{F}}$ *concordant to* j*. In other words, concordance classes of codimension* q *foliations on* M *correspond bijectively to concordance classes of* Γ_q*-structure on* M *together with concordance classes of bundle monomorphisms of their normal bundles into* $T(M)$*.*

Here we consider $\nu_{\mathcal{F}}$ as a subbundle of $T(M)$ and identify $\nu_{\mathcal{F}}$ with ν_γ by using the normal bundle over $M \times I$ of the concordance between γ and $\mathcal{F}$. This defines a unique homotopy class of bundle isomorphisms $\nu_{\mathcal{F}} \to \nu_\gamma$.

If $q > 1$ there is a relative version, when γ is already a foliation in a neighborhood of a closed subset A of M. Then the foliation $\mathcal{F}$ can be made to agree with γ in a possibly smaller neighborhood of A. In the words of Y. Eliashberg, Γ_q-structure with $q > 1$ satisfy the h-principle.

For $q = 1$ the relative version does not hold. By the Reeb stability theorem, if $\mathcal{F}$ is a transversely orientable foliation of codimension 1 of a closed manifold containing a single simply connected leaf F, then M is a bundle over S^1 with fibre F. So if $M^3 = S^2 \times S^1 \# N$ is a connected sum where $N \neq S^3$. Let $S = S^2 \times \{t\}$ be not affected by forming the connected sum. There is a line field transverse to S and we can foliate a neighborhood of S by parallel $2 - spheres$. By Reeb's theorem this foliation does not extend to M.

On pages 220 and 221 of [Th2] there is a nice outline of the proof of the theorem in codimension greater than 1, but the details are geometrically quite involved. There is another proof by Y. Eliashberg and N. M. Mishachev [EM] using singularity theory.

Theorem 11 puts the topology of the homotopy fibre of the normal bundle map $\nu : B\Gamma_q \to BGl_q(\mathbb{R})$ into the limelight. The only obstruction to finding for a given q-dimensional subbundle μ of $T(M)$ a foliation whose normal bundle is concordant to μ is the obstruction to lifting the classifying map $f_\mu : M \to BGl_q(\mathbb{R})$ of μ to $B\Gamma_q$. The obstructions for lifting lie in the cohomology groups of M with coefficients in the homotopy groups of the homotopy fiber of ν. This space is usually denoted by $B\bar{\Gamma}_q$ and is defined as follows.

Denote by $P(\nu)$ the subspace of $B\Gamma_q \times BGl_q(\mathbb{R})^I$ of all pairs (x, ω) with $\nu(x) = \omega(0)$. Here $BGl_q(\mathbb{R})^I$ is the space of paths $\omega : I \to BGl_q(\mathbb{R})$ with the compact open topology. The projection $P(\nu) \to B\Gamma_q$ is a homotopy equivalence and the map $p(\nu) : P(\nu) \to BGl_q(\mathbb{R})$, $(x, \omega) \mapsto \omega(1)$, is a Hurewicz fibration, i. e. a map having the homotopy lifting property for all spaces. The homotopy fibre of ν is by definition the inverse image $p(\nu)^{-1}(*)$ of the base point $*$ of $BGl_q(\mathbb{R})$. Which base point we choose is immeterial since $BGl_q(\mathbb{R})$ is path connected, but the equivalence class of (id, e_0) is the usual choice.

Thus a map $\bar{f} : M \to B\bar{\Gamma}_q$ is a map $f : M \to B\Gamma_q$ together with a homotopy $h : M \times I \to BGl_q(\mathbb{R})$ from $\nu \circ f$ to the constant map. Such a homotopy defines a bundle isomorphism from the normal bundle of the Γ_q-structure associated to f with the trivial q-dimensional vectorbundle over M, and this bundle isomorphism is unique up to homotopy. Thus $B\bar{\Gamma}_q$ classifies Γ_q-structure structures with a homotopy class of trivialisations of their normal bundles.

The obstruction for extending a lift of a map $f : X \to BGl_q(\mathbb{R})$ to $B\Gamma_q$ given on the $(k-1)$-skeleton of the CW-complex X to the k-skeleton, after a possible change of the lift on the $(k-1)$-cells, is an element of $H^k(M; \pi_{k-1}(B\bar{\Gamma}_q))$.

If one is interested in foliations of codimension q on an n-manifold M, then n should at least be q, and if $n = q$ or $q + 1$ we know that any $n - q$- field is integrable. So there always will be a lift to $B\Gamma_q$ of the map into $BGl_q(\mathbb{R})$ classifying the complementary q-plane field. In fact the corresponding obstruction groups to lifting vanish, as was proved by Haefliger [Hae4]

1.12 Theorem. *For all $1 \le r \le \infty$ the space $B\bar{\Gamma}^r_q$ is q-connected.*

In fact, this also holds for $r = 0$ by using the appropriate concepts in the topological category, where one uses microbundles from the outset. As before, the decoration r refers to the differentiabilty of the local diffeomorphisms of $\mathbb{R}^q$ and thus to the transverse differentiability of the foliations under consideration.

Our knowledge of the vanishing of the low dimensional homotopy groups of $B\bar{\Gamma}^r_q$ depends strongly on r. Here are some of the relevant results.

1.13 Theorem.

(i) For $r = 0$ or $r = 1$ the space $B\bar{\Gamma}^r_q$ is contractible.

(ii) For $r \neq q + 1$ the space $B\bar{\Gamma}^r_q$ is $(q+1)$-connected.

(iii) For $r < \lfloor q/2 \rfloor$ the space $B\bar{\Gamma}^r_q$ is $(q+2)$-connected.

(iv) For $r \le \lfloor (q+1)/m \rfloor - 1$ the space $B\bar{\Gamma}^r_q$ is $(q+m)$-connected.

Part (i) is due to Mather [Ma1] and Tsuboi [Tsu2] who have to evoke a theorem, due in codimension 1 to Mather [Ma 2] and in general to Thurston [Th1] with proofs published in [Ma 4], [McD1], [McD2], which relates the homology groups of the group $B\overline{Diff^r_c}\mathbb{R}^q$ to the homology of the q-fold loop space $\Omega^q B\bar{\Gamma}_q$ of $B\bar{\Gamma}_q$. Here $\overline{Diff^r_c}\mathbb{R}^q$ is the homotopy fibre of the map $Diff^{r\ \delta}_c\mathbb{R}^q \to Diff^r_c\mathbb{R}^q$ where $Diff^r_c\mathbb{R}^q$ is the group of C^r-diffeomorphisms of $\mathbb{R}^q$ with compact support and its usual C^r-topology, and $Diff_c^{r\ \delta}\mathbb{R}^q$ is the same group with the discrete topology. We will indicate a proof of this so-called Thurston–Mather theorem in the codimension 1 case in the next section using as in the proof of [McD1]and [McD2] ideas of Graeme Segal (see [Se3]).

Part (ii) is in codimension 1 due to Mather [Ma 3] and in general to Thurston [Th1], and parts(iii) and (iv) are due to Tsuboi [Tsu1].

So the connectivity, as known to us today, increases when the differentiability goes down. It is in the meantime a long standing conjecture with absolutely no progress within the last 35 years that for all $r \leq \infty$ the space $B\bar{\Gamma}_q$ is $2q$-connected.

Thus with regard to the question whether any given $(n-q)$-plane field on a closed n-manifold M is homotopic to the tangent plane field of a smooth foliation, the only new thing this theorem tells us is that the answer is yes if $n-q$ equals 2. Should the conjecture hold then we get a positive answer for all $q \geq (n-1)/2$.

The high dimensional homotopy groups of $B\bar{\Gamma}_q$ have a tendency to be enormous. We will briefly state some of the known facts, but will not explain any of the details. Most of these results are derived from our knowledge of the non-vanishing and variation of some of the secondary characteristic classes for foliations. These are real cohomology classes with real coefficients which ultimately live in the real cohomology rings of $B\Gamma_q$ and $B\bar{\Gamma}_q$. These classes owe their existence to a wonderful observation of R. Bott.

1.14 Theorem. **(Bott vanishing Theorem)** *Let ν be the normal bundle of a C^2-foliation of codimension q on the manifold M. Then all elements in the subring of $H^*(M;\mathbb{R})$ generated by the Pontryagin classes of ν vanish in degrees greater than $2q$.*

The theorem is easy to prove using the Chern-Weil theory of characteristic classes via connections and curvature forms. It gives rise to two graded commutative differential algebras $WO_q = \Lambda(h_1, h_3, \ldots, h_{2l-1}, c_1, c_2, \ldots, c_q)/I_q$ and $W_q = \Lambda(h_1, h_2, \ldots, h_q, c_1, c_2, \ldots, c_q)/I_q$, where l is the largest integer such that $2l-1 \leq q$. The c_i have degree $2i$ and are cocycles, the h_i have degree $2i-1$ with differential $d(h_i) = c_i$. In both cases I_q is the ideal generated by all products of the c_i of degree greater than $2q$. Without factoring out I_q the associated cohomology rings would be the one of the subring of $H^*(BGl_q(\mathbb{R}), \mathbb{R})$ generated by the universal Pontryagin classes in the case of WO_q, and the other one would be trivial, which corresponds to the Pontryagin algebra of the trivial bundle. The Bott vanishing theorem allows us to quotient out I_q when dealing with normal bundles of foliations.

The construction is natural with regard to foliation preserving smooth maps and well defined after passing to cohomology. Thus one obtains for $r \geq 2$ well defined maps

$$ch : H^*(WO_q) \to H^*(B\Gamma_q^r; \mathbb{R})$$
$$ch : H^*(W_q) \to H^*(B\bar{\Gamma}_q^r; \mathbb{R})$$

By constructing explicit examples of foliations, quite a few of these classes were shown to be linearly independent, and some of them even vary independently, i. e. there are examples of k-tuples of classes $(x_1, \ldots, x_k)$ of the same degree p in $H^*(WO_q)$ such that the evaluation map

$$(ch(x_1), \ldots, ch(x_k)) : H_p(B\Gamma_q; \mathbb{Z}) \to \mathbb{R}^k$$

is surjective, and similarly for $H^*(W_q)$ and $H_*(B\bar{\Gamma}_q^r; \mathbb{Z})$.

These results are due to many people, notably Thurston, Heitsch, Kamber and Tondeur, Baker, Hurder.

We should also mention that it is a long standing conjecture that the two maps *ch* are injective.

From the results just mentioned and the fact that all products in $H^*(WO_q)$ and $H^*(W_q)$ vanish, Hurder [Hu2] shows among many other things

1.15 Theorem. *There are epimorphisms*

$$\pi_i(B\overline{\Gamma}_q^r) \longrightarrow \mathbb{R}^{v_{q,i}}$$

with $\limsup_{i\to\infty} v_{q,i} = \infty$ *for any* $q \geq 2,\ r \geq 2$.

1.16 Remark. The bibliography below contains items that have not been referred to in this introductory text. They will be used in future notes which cover other talks at the BGamma School 2018.

References

[BSS] **R Bott**, **H Shulman**, **J Stasheff**, *On the deRham theory of certain classifying spaces*, Advances of Math. 20 (1976), 43–56

[BuL] **J P Buffet**, **J C Lor**, *Une construction d'un universel pour une classe assez large de* Γ*-structures*, C. R. Acad. Sc. Paris 270 (1970), 640-642

[tD1] **T tom Dieck**, *On the homotopy type of classifying spaces*, manuscripta math. 11 (1974), 41–49

[tD2] **T tom Dieck**, *Algebraic Topology*, EMS Textbooks in Mathematics, European Math. Soc. Publishing House, Z"urich, 2008

[EM] **Y Eliashberg**, **N M Mishachev**, *Wrinklings of smooth mappings III; Foliations of codimension greater than one*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center 11, 1998, 321350

[God] **C Godbillon**, *Cohomologies d'algèbres de Lie de champs de vecteurs formels*, Séminaire Bourbaki 25e année, (1972/73), n^o 421, 69-87

[GF] **I M Gelfand**, **D B Fuks**, *Cohomology of the Lie algebra of formal vector fields*, Izv. Akad. Nauk SSSR. 34 (1975) 322–337

[Hae1] **A Haefliger**, *Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comm. Math. Helv. 32 (1958) 248–329

[Hae2] **A Haefliger**, *Cohomology of Lie Algebras and Foliations*, In: Differential Topology, Foliations and Gelfand-Fuks Cohomology, Proceedings, Rio de Janeiro, 1976. Lecture Notes in Math. 652, 1-12. Springer Verlag 1978.

[Hae3] **A Haefliger**, *Groupoides d'holonomie et classifiants*, Structure Transverse des Feuilletages, Toulouse 1982, Asterisque, vol. 116, Soc. Math. de France 1984, 70–97

[Hae4] **A Haefliger**, *Homotopy and integrability*, Manifolds– Amsterdam 1970, Proc. of the Nuffic Summer School on Manifolds, Amsterdam, August 17–29, 1970, Lect. Notes in Math. 197, Springer 1971, 133–163

[Hu1] **S Hurder**, *Dual Homotopy Invariants of G-Foliations,* Topology 20 (1981), 365–387

[Hu2] **S Hurder**, *The Classifying Space of Smooth Foliations,* Illinois J. Math. 29 (1985), 108–133

[Ma1] **J Mather**, *The vanishing of the homology of certain groups of homeomorphisms*, Topology 10 (1971), 297–298

[Ma 2] **J Mather**, *Integrability in codimension 1.* Comment. Math. Helv. 48 (1974), 195–233

[Ma 3] **J Mather**, *Commutators of Diffeomorphisms I and II.* Comment. Math. Helv. 49 (1974), 512-528, and 50 (1975), 33-40

[Ma 4] **J Mather**, *On the Homology of Haefliger's Classifying Space.* C.I.M.E., Differential Topology (1976), 71-116

[McD1] **D McDuff**, *Foliations and monoids of embeddings*, in *Geometric Topology*, ed. Cantrell, Academic Press (1979), 429–444

[McD2] **D McDuff**, *The homology of some groups of diffeomorphisms*, Comm. Math. Helv. 55 (1980), 97–129

[McDSe] **D McDuff**, **G Segal**, *Homology fibrations and the "group-completion" theorem*, Invent. Math. 31 (1976), 279–284

[Me] **G Meigniez**, *Fibrations, submersions and bundles*, Trans. Amer. Math. Soc. 354 (2002), 3771-3787

[Se1] **G Segal**, *Classifying spaces and spectral sequences*, Publ. Math. Inst. Hautes Etudes Sc. 34 (1968), 105–112

[Se2] **G Segal** *Categories and cohomology theories*, Topology 13 (1974) , 293–312

[Se3] **G Segal** *Classifying spaces related to foliations*, Topology 17 (1978) , 367–382

[St] **J Stasheff**, *Construction of BC, Appendix B to: R Bott, Lectures on characteristic classes and foliations*, Lect. Notes in Math. 279, 1–94, Springer 1972

[Su] **D Sullivan**, *Infinitesimal Computations in Topology*, Publ. Math. I.H.E.S. 47 (1977), 269-331

[Th1] **W Thurston**, *Foliations and groups of diffeomorphisms*, Bull AMS 80 (1974), 304–307

[Th2] **W Thurston**, *The theory of foliations of codimension greater than one*, Comm. Math. Helv. 49 (1974), 214–231

[Th3] **W Thurston**, *Existence of codimension-one foliations*, Ann. Math. 104 (1976), 249–268

[Tsu1] **T Tsuboi**, *On the Homology of Classifying Spaces of Foliated Products*. Advanced Studies in Pure Mathematics 5 (1985), 37-120

[Tsu2] **T Tsuboi**, *On the foliated products of class C^1*, Ann. Math. 130 (1989), 227–271

[Vo] **R M Vogt**, *Convenient categories of topological spaces for homotopy theory*, Arch. Math. 22 (1971), 545–555