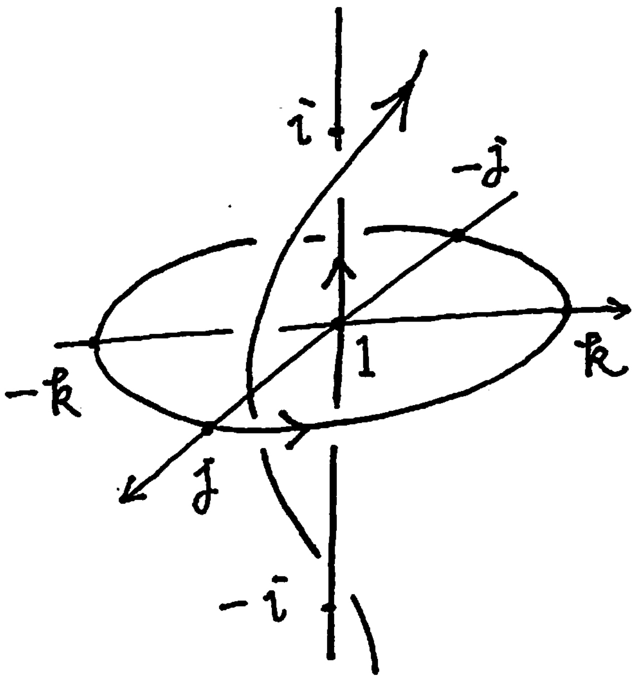
Hopf fibration

$$S^1 \times S^3 \longrightarrow S^3$$

$$(\omega, q) \longmapsto \omega q$$

$$\shortparallel$$

$$(\omega, z_1 + z_2 j) \mapsto \omega z_1 + \omega z_2 j$$

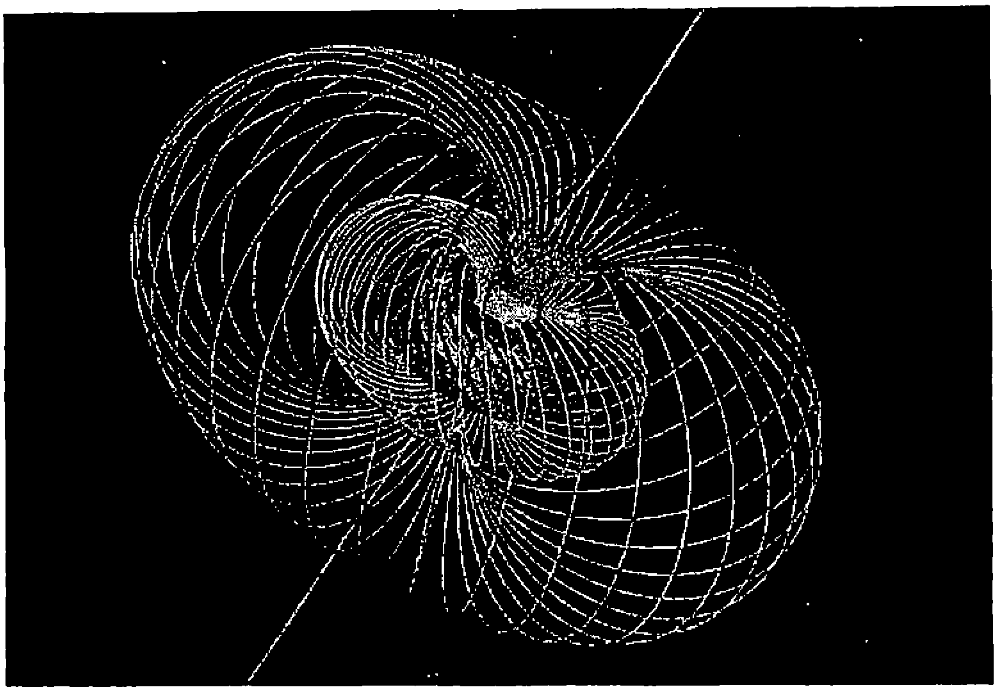


$$h : S^3 \longrightarrow S^2$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ q & \longmapsto & q^{-1} i q \end{array}$$

Then

$$\begin{tikzcd} S^3 \arrow[r, "\mathrm{Proj}"] \arrow[dr, "h"'] & S^1 \backslash S^3 \arrow[d, phantom, "\cong"] \\ & S^2 \end{tikzcd}$$



- **Def** Poincare homology sphere $PH := S^3/\phi(1\times I^*)$
$$= S^3/I^*$$

- Right action of S^3 on S^3 preserves the Hopf fibration

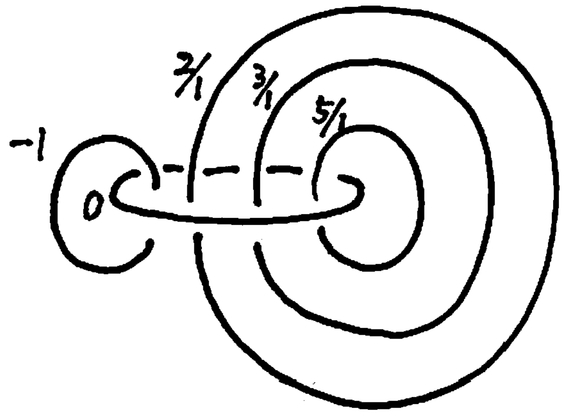
$$\begin{array}{ccc} S^3 & \xrightarrow{\phi(1,q)} & S^3 \\ h\downarrow & & \downarrow h \\ S^2 & \xrightarrow{\psi(q)} & S^2 \end{array}$$

- **Prop** PH is an S^1-bundle over the orbifold $S^2(2,3,5)$

(∵)
$$\begin{array}{ccl} S^3 & \longrightarrow & PH = S^3/\phi(1\times I^*) \\ h\downarrow & & \downarrow \\ S^2 & \longrightarrow & S^2/\psi(I^*) \cong S^2/I \cong S^2(2,3,5) \end{array}$$

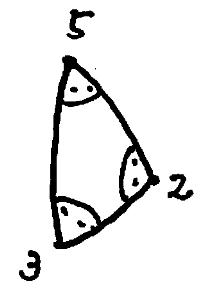
Prop PH is a Seifert fibered space over the orbifold $S^2(2,3,5)$.

$$PH \cong \{-1\,;\,(2,1),(3,1),(5,1)\}$$



(Proof)

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, "h"'] & PH = S^3/\Phi(1\times I^*) \arrow[d] \\ S^2 \arrow[r] & S^2/\Psi(I^*) \end{tikzcd} \cong S^2/I \cong S^2(2,3,5) \cong$$



singular fiber の <u>Seifert invariant</u> と

(normal) <u>Euler number</u> を求めれば良い.

- Seifert fibered space is an "S^1-bundle over a 2-dim orbifold"

  ie 3-manifold "foliated" by circles, st

  each circle has a fibered nbd with orbit invariant (p, q).

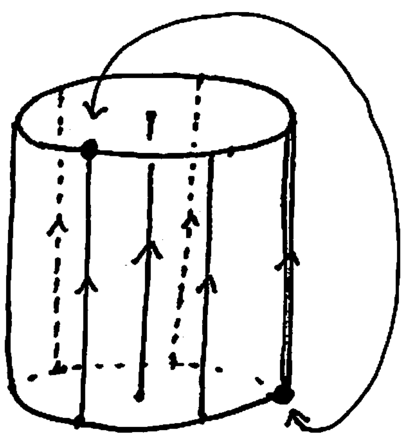
  (p, q are mutually prime integers)

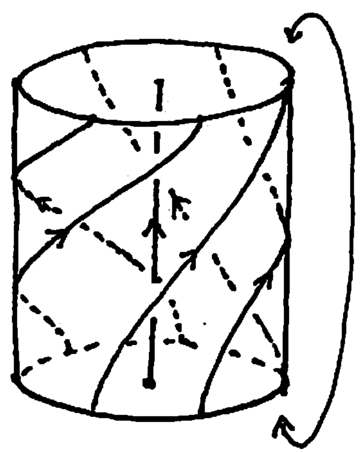
Fibered nbd of orbit invariant (p, q)

$$:= D^2 \times S^1 / \langle g \rangle , \quad g := \left(\left(\frac{-q}{p}\right) 回転 \right) \times \left(\frac{1}{p} 回転 \right) : D^2 \times S^1 \to D^2 \times S^1$$

$$\cong D^2 \times [0, 1] \Big/ (z, 0) \sim (e^{2\pi i \left(\frac{-q}{p}\right)} z, 1)$$

$(p, q) = (5, 2)$





(Regular fiber)

$= p$ (singular fiber)

$+ q$ (meridian)

- Seifert invariant of a singular fiber with orbit inv. (p, q)

  is (α, β), where $\alpha = p$, $\beta \equiv q^{-1} \mod \alpha = p$.

<u>Fact</u> F: compact surface with $\partial F = S^1$

$$D^2 \times S^1 \supset \partial D^2 \times S^1 \xrightarrow[\cong]{\varphi} \partial F \times S^1 \subset F \times S^1$$

$$[\partial D^2] \longmapsto \alpha[\partial F] + \beta[S^1]$$

Then $M := D^2 \times S^1 \cup_{\varphi} F \times S^1$ is a Seifert fibered space, and

$0 \times S^1$ is a (singular) fiber with Seifert invariant (α, β)

orbit invariant (p, q)

where $p = \alpha$, $q \equiv \beta^{-1} \ (\alpha)$.

Dehn surgery

$K \subset S^3$ (oriented) knot

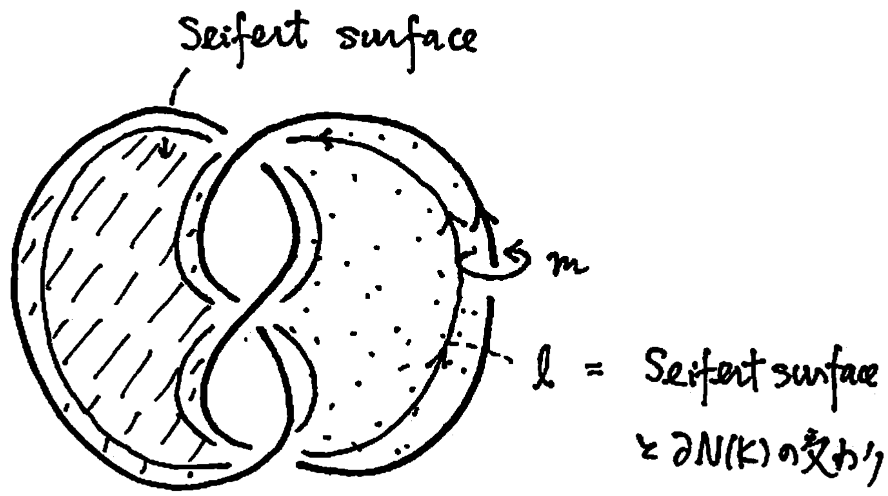
$N(K)$: regular nbd of K

$E(K) := S^3 - \mathring{N}(K)$: exterior

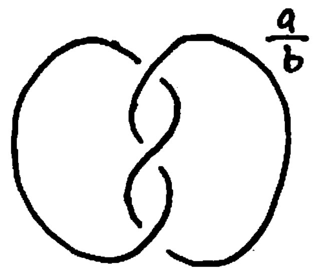
$\cup$

$\partial E(K) \supset l$ longitude $lk(K, l) = 0$, $K \sim l$ in $N(K)$

m meridian $lk(K. m) = +1$



$\frac{a}{b}$ - surgery on $K := D^2 \times S^1 \underset{\varphi}{\cup} E(K)$



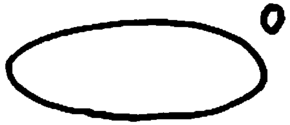
但し $\varphi : \partial D^2 \times S^1 \longrightarrow \partial E(K)$

$[\partial D^2] \longmapsto a[m] + b[l]$

<u>Remark</u> $K(\frac{a}{b}) = (E(K) \underset{\varphi}{\cup} D^2 \times I) \underset{S^2}{\cup} (\underbrace{D^2 \times S^1 - D^2 \times I}_{\substack{\cong \\ B^3}})$ なので

$\frac{a}{b}$ で決まる。(φは一意でないが…)

Example



$\cong S^2 \times S^1$: Seifert fibered space over S^2

($\because$) $E(\text{trivial knot}) \cong D^2 \times S^1$

$m \longleftrightarrow * \times S^1$

$l \longleftrightarrow \partial D^2 \times *$

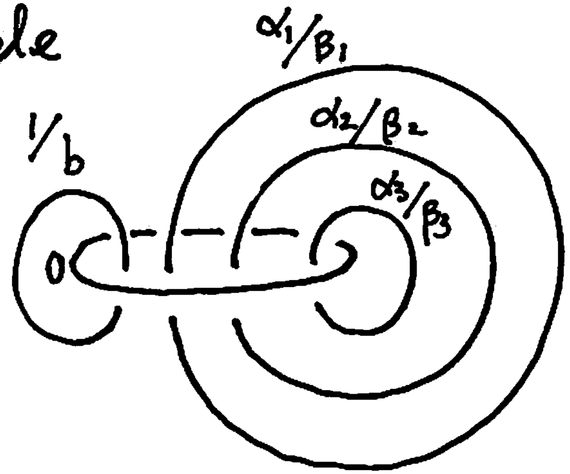
$\varphi : \partial D^2 \times S^1 \xrightarrow[\cong]{} \partial E(K) = \partial(D^2 \times S^1)$

$[\partial D^2] \longmapsto 0\,m + 1 \cdot l = \partial D^2$

$\therefore$ 

$\cong D^2 \times S^1 \cup_{id} D^2 \times S^1 \cong S^2 \times S^1$

Example



$\cong$ Seifert fibered space

$\{ b ; (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r) \}$

over $S^2(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Normal Euler number

$$e_0 := -\left\{ b + \frac{\beta_1}{\alpha_1} + \cdots + \frac{\beta_r}{\alpha_r} \right\} \in \mathbb{Q} \qquad (b \in \mathbb{Z})$$

Prop If $r \geqq 3$, then

the multiple set $\left\{ \frac{\beta_i}{\alpha_i} \in \mathbb{Q}/\mathbb{Z} \;\middle|\; 1 \leq i \leq r \right\}$ and

the Euler number $e_0 = -\left\{ b + \sum_i \frac{\beta_i}{\alpha_i} \right\}$

form the complete invariant of the oriented mfd

$\{ b ; (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r) \}$

Covering formula for Euler number

$$\begin{array}{ccc} \tilde{M}^3 & \xrightarrow{\tilde{p}} & M^3 \\ \tilde{\eta} \downarrow & & \downarrow \eta \\ \tilde{O}^2 & \xrightarrow{p} & O^2 \end{array}$$

$\tilde{d} := \deg(\tilde{p} : \tilde{M}^3 \to M^3)$

$d := \deg(p : \tilde{O}^2 \to O^2)$

$m := \tilde{d}/d$ = deg for fiber

$$\Rightarrow e_0(\eta) = \frac{m}{d} e(\tilde{\eta})$$

$PH = S^3/\Phi(1\times I^*)$ の Seifert invariant と Euler number の計算

- Singular fiber の Seifert invariants

Sakuma : The geometries of spherical Montesinos links

Kobe J. Math 7 (1990) 167-190

Lemma 1.1 + Lemma 2.9 参照

(お詫び) 上記論文には「恥かしいミス」があります.

orbit invariant と Seifert invariant の混同

- $$\begin{tikzcd} S^3 \arrow[r, "\tilde{p}"] \arrow[d, "h"'] & S^3/\Phi(1\times I^*) \arrow[d, "\eta"] \\ S^2 \arrow[r, "p"] & S^2(2,3,5) \end{tikzcd}$$

$\tilde{d} = \deg \tilde{p} = |I^*| = 120$

$d = \deg p = |I| = 60$

$m = 120/60 = 2$

$$e_0(\eta) = \frac{m}{d}\, e(h) = \frac{2}{60}\times(-1) = \frac{-1}{30} = -\left\{ b + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \right\} \quad \therefore b = -1$$