正20面体群からの旅たちII

東京農工大学
関口次郎

§2. 17種類の多項式

x, y, z を変数，p, q, r を $p < q < r$ が成り立つ自然数
p, q, r は共約数を持たないと仮定する．
(x, y, z) 空間上の次のような3個のベクトル場 V^0, V^1, V^2 を定義する：

$$\begin{cases} V^0 &= px\dfrac{\partial}{\partial x} + qy\dfrac{\partial}{\partial y} + rz\dfrac{\partial}{\partial z}, \\ V^1 &= qy\dfrac{\partial}{\partial x} + h_{22}(x,y,z)\dfrac{\partial}{\partial y} + h_{23}(x,y,z)\dfrac{\partial}{\partial z}, \\ V^2 &= rz\dfrac{\partial}{\partial x} + h_{32}(x,y,z)\dfrac{\partial}{\partial y} + h_{33}(x,y,z)\dfrac{\partial}{\partial z}, \end{cases} \tag{1}$$

$h_{ij}(x, y, z)$ は x, y, z の多項式
また V^0, V^1, V^2 から 3×3 行列 M を以下のようにして定義する：

$$M = \begin{pmatrix} px & qy & rz \\ qy & h_{22}(x,y,z) & h_{23}(x,y,z) \\ rz & h_{32}(x,y,z) & h_{33}(x,y,z) \end{pmatrix}. \tag{2}$$

x, y, z の関数 $f(x, y, z)$ が (p, q, r) 型重みつき斉次多項式であるとは，適当な自然数 d に対して $V_0 f = df$ が成り立つことである．このとき，d は f の次数という．

V_0, V_1, V_2 に対する条件を考える：

条件 2.1

(i) $[V_0, V_1] = (q-p)V_1$, $[V_0, V_2] = (r-p)V_2$.

(ii) 次を満たす多項式 $f_j(x,y,z)$ $(j=0,1,2)$ が存在する：
$[V_1, V_2] = f_0(x,y,z)V_0 + f_1(x,y,z)V_1 + f_2(x,y,z)V_2$.

(iii) $\dfrac{\partial h_{22}}{\partial z}$ は零でない定数

(iv) 多項式 $\det(M)$ は z^3 の定数倍でない.

問題 2.2 上の条件 2.1 を満たすベクトル場の三つ組 $\{V_0, V_1, V_2\}$ を，重みつき座標変換を除いて分類せよ.

V_0, V_1, V_2 が条件 2.1 を満たし，$\det(M)$ が重複因子をもたない多項式のとき，$\det(M) = 0$ で定まる超曲面は対数的自由因子になる.

問題 2.2 に対する解答は次の定理であたえられる．

定理 2.3 x, y, z を変数，p, q, r を $p < q < r$ であるような自然数とする．p, q, r は共約数をもたないとする．このとき，次が成り立つ．

(i) もし $(p, q, r) \neq (2, 3, 4)$, $(1, 2, 3)$, $(1, 3, 5)$ であれば，条件を満たすベクトル場の三つ組 $\{V_0, V_1, V_2\}$ は存在しない．

(ii) もし (p, q, r) が $(2, 3, 4)$, $(1, 2, 3)$, $(1, 3, 5)$ のいずれかであれば，$F = \det(M)$ の形の多項式 $F(x, y, z)$ は重みつき座標変換によって次のいずれかと一致するようにできる：

(ii.1) $(p,q,r)=(2,3,4)$ の場合 (この場合は A_3 型鏡映群に対応する)

$F_{A,1}=16x^4z-4x^3y^2-128x^2z^2+144xy^2z-27y^4+256z^3$.

$F_{A,2}=2x^6-3x^4z+18x^3y^2-18xy^2z+27y^4+z^3$.

(ii.2) $(p,q,r)=(1,2,3)$ の場合 (この場合は B_3 型鏡映群に対応する)

$F_{B,1}=z(x^2y^2-4y^3-4x^3z+18xyz-27z^2)$.

$F_{B,2}=z(-2y^3+4x^3z+18xyz+27z^2)$.

$F_{B,3}=z(-2y^3+9xyz+45z^2)$.

$F_{B,4}=z(9x^2y^2-4y^3+18xyz+9z^2)$.

$F_{B,5}=xy^4+y^3z+z^3$.

$F_{B,6}=9xy^4+6x^2y^2z-4y^3z+x^3z^2-12xyz^2+4z^3$.

$F_{B,7}=\frac{1}{2}xy^4-2x^2y^2z-y^3z+2x^3z^2+2xyz^2+z^3$.

(ii.3) $(p, q, r) = (1, 3, 5)$ の場合 (この場合は H_3 型鏡映群に対応する)

$F_{H,1} = -50z^3 + (4x^5 - 50x^2y)z^2 + (4x^7y + 60x^4y^2 + 225xy^3)z - \frac{135}{2}y^5 - 115x^3y^4 - 10x^6y^3 - 4x^9y^2.$

$F_{H,2} = 100x^3y^4 + y^5 + 40x^4y^2z - 10xy^3z + 4x^5z^2 - 15x^2yz^2 + z^3.$

$F_{H,3} = 8x^3y^4 + 108y^5 - 36xy^3z - x^2yz^2 + 4z^3.$

$F_{H,4} = y^5 - 2xy^3z + x^2yz^2 + z^3.$

$F_{H,5} = x^3y^4 - y^5 + 3xy^3z + z^3.$

$F_{H,6} = x^3y^4 + y^5 - 2x^4y^2z - 4xy^3z + x^5z^2 + 3x^2yz^2 + z^3.$

$F_{H,7} = xy^3z + y^5 + z^3.$

$F_{H,8} = x^3y^4 + y^5 - 8x^4y^2z - 7xy^3z + 16x^5z^2 + 12x^2yz^2 + z^3.$

$F_{A,1}$	$A_2 + A_2 + A_1$
$F_{A,2}$	A_5
$F_{B,1}$	$A_3 + A_2 + A_1$
$F_{B,2}$	$A_5 + A_1$
$F_{B,3}$	D_6
$F_{B,4}$	$D_5 + A_1$
$F_{B,5}$	E_6
$F_{B,6}$	$A_4 + A_2$
$F_{B,7}$	A_6
$F_{H,1}$	$A_4 + A_2 + A_1$
$F_{H,2}$	$A_4 + A_3$
$F_{H,3}$	$D_5 + A_2$
$F_{H,4}$	D_7
$F_{H,5}$	$E_6 + A_1$
$F_{H,6}$	A_7
$F_{H,7}$	E_7
$F_{H,8}$	$A_6 + A_1$

注意 2.4 ここに，多項式 $F_{A,1},\dots,F_{H,8}$ それぞれの場合に対応する行列 M を与えておく．(x1,x2,x3 は x,y,z のことである．)

```
MF_{A,1} = {{2*x1,3*x2, 4*x3},
            {3*x2, -x1^2 + 4*x3, -1/2*x1*x2},
            {4*x3, -1/2*x1*x2, 1/4*(8*x1*x3 - 3*x2^2)}}
MF_{A,2} = {{2*x1,3*x2, 4*x3},
            {3*x2, 1/2*(x3 - x1^2),6*x1*x2},
            {4*x3, -2*x1*x2, 16*x1^3 + 24*x2^2 - 8*x1*x3}}

MF_{B,1} = {{x1, 2*x2, 3*x3},
            {2*x2, x1*x2 + 3*x3, 2*x1*x3},
            {3*x3, 2*x1*x3, x2*x3}}
MF_{B,2} = {{x1, 2*x2, 3*x3},
            {2*x2, -2/3*(2*x1*x2 - 9*x3), -4*x1*x3},
            {3*x3, -2/3*(x2^2 + 3*x1*x3), -2*x2*x3}}
MF_{B,3} = {{x1, 2*x2, 3*x3},
            {2*x2, -3/5*(x1*x2 - 5*x3), -6/5*x1*x3},
            {3*x3, -3/5*x2^2, -6/5*x2*x3}}
MF_{B,4} = {{x1, 2*x2, 3*x3},
            {2*x2, 3(3*x1*x2 + x3), 6*x1*x3},
            {3*x3, 0, -3*x2*x3}}
MF_{B,5} = {{x1, 2*x2, 3*x3},
            {2*x2, -24*x1*x2 + 2*x3, -2*x2^2 - 32*x1*x3},
            {3*x3, -9*x2^2, -12*x2*x3}}
MF_{B,6} = {{x1, 2*x2, 3*x3},
            {2*x2, 3*x1*x2 + 5/2*x3, 9/2*x2^2 + 15/2*x1*x3},
            {3*x3,3/4*(15*x2^2 + x1*x3), 18*x2*x3}}
MF_{B,7} = {{x1, 2*x2, 3*x3},
            {2*x2, 1/3*(-4*x1*x2 + 7*x3), x2^2 - 14/3*x1*x3},
            {3*x3, 3/2*(7*x2^2 - 6*x1*x3), 12*x2*x3}}
```

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zsa
MF_{H,1} = {{x1, 3*x2, 5*x3},
            {3*x2, 2*x3 + 2*x1^2*x2, 7*x1*x2^2 + 2*x1^4*x2},
            {5*x3, 7*x1*x2^2 + 2*x1^4*x2,
             1/2*(15*x2^3 + 4*x1^4*x3 + 18*x1^3*x2^2)}}
MF_{H,2} = {{x1, 3*x2, 5*x3},
            {3*x2, 36*x1^2*x2 + 6*x3, 90*x1*x2^2 + 90*x1^2*x3},
            {5*x3, -10/3*(12*x1^3 - 55*x2)*x1*x2,
             -50/3*(6*x1^3*x2^2 - x2^3 + 6*x1^4*x3 - 18*x1*x2*x3)}}
MF_{H,3} = {{x1, 3*x2, 5*x3},
            {3*x2, 1/10*(x1^2*x2 + 2*x3), 23/10*x1*x2^2 + 3/20*x1^2*x3},
            {5*x3, 5*x1*x2^2, 15/2*x2(2*x2^2 + x1*x3)}}
MF_{H,4} = {{x1, 3*x2, 5*x3},
            {3*x2, 1/5*(-4*x1^2*x2 + 6*x3), 2/5*x1*x2^2 - 2*x1^2*x3},
            {5*x3, -20/3*x1*x2^2,10/3*x2*(x2^2 - 5*x1*x3)}}
MF_{H,5} = {{x1, 3*x2, 5*x3},
            {3*x2, -9/5*(4*x1^2*x2 - x3), -3/5*x1(9*x2^2 + 16*x1*x3)},
            {5*x3, -15*x1*x2^2, -5*x2(x2^2 + 4*x1*x3)}}
MF_{H,6} = {{x1, 3*x2, 5*x3},
            {3*x2, -3/5*(3*x1^2*x2 - 4*x3), -18/5*x1*(-x2^2 + 2*x1*x3)},
            {5*x3, -5/3*x1*(-8*x2^2 + 5*x1*x3), 10/3*x2*(2*x2^2 + x1*x3)}}
MF_{H,7} = {{x1, 3*x2, 5*x3},
            {3*x2, -3/5*(2*x1^2*x2 + x3), -3/5*x1(-x2^2 + 3*x1*x3)},
            {5*x3, 10/3*x1*x2^2, -5/3*x2*(x2^2 - 3*x1*x3)}}
MF_{H,8} = {{x1, 3*x2, 5*x3},
            {3*x2, -3/5*(24*x1^2*x2 - 7*x3), -9/5*x1(-3*x2^2 + 28*x1*x3)},
            {5*x3, -5/3*x1*(7*x2^2 + 20*x1*x3), 5/3*x2*(7*x2^2 - 52*x1*x3)}}
```

§. 一般化

x, y, z を変数，p, q, r を $p < q < r$ が成り立つ自然数

p, q, r は共約数を持たないと仮定する．

(x, y, z) 空間上の次のような3個のベクトル場 V^0, V^1, V^2 を定義する：

$$\begin{cases} V^0 = px\frac{\partial}{\partial x} + qy\frac{\partial}{\partial y} + rz\frac{\partial}{\partial z}, \\ V^1 = h_{21}\frac{\partial}{\partial x} + h_{22}\frac{\partial}{\partial y} + h_{23}\frac{\partial}{\partial z}, \\ V^2 = h_{31}\frac{\partial}{\partial x} + h_{32}\frac{\partial}{\partial y} + h_{33}\frac{\partial}{\partial z}, \end{cases} \tag{3}$$

$h_{ij} = h_{ij}(x, y, z)$ は x, y, z の多項式

また V^0, V^1, V^2 から 3×3 行列 M を以下のようにして定義する：

$$M = \begin{pmatrix} px & qy & rz \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \tag{4}$$

V_0, V_1, V_2 に対する条件を考える：

条件 2.1'

(i) $[V_0, V_1] = s_1 V_1,\ [V_0, V_2] = s_2 V_2 \quad (s_1, s_2$: 定数)

(ii) 次を満たす多項式 $f_j(x, y, z)$ $(j = 0, 1, 2)$ が存在する：
$[V_1, V_2] = f_0(x, y, z)V_0 + f_1(x, y, z)V_1 + f_2(x, y, z)V_2.$

問題 2.2' 上の条件 2.1' を満たすベクトル場の三つ組 $\{V_0, V_1, V_2\}$ を，重みつき座標変換を除いて分類せよ.

V_0, V_1, V_2 が条件 2.1' を満たし，$\det(M)$ が重複因子をもたない多項式のとき，$\det(M) = 0$ で定まる超曲面は対数的自由因子になる.

複素鏡映群の判別式はこのような条件を満たす.
(定理ではなく観察)

§6. 階数 3 の既約な鏡映群

ここで扱うのは，A_3, B_3, H_3 型実鏡映群と [ST] で

No.24,No.25,No.26,No.27

と番号のついた複素鏡映群である．H_3 型実鏡映群は [ST] では No.23 になっている．G をこれらの群のいずれかとする．

このとき，P_1, P_2, P_3 を代数的に独立な G-不変斉次多項式環の生成元とする．$k_j = \deg_\xi(P_j)$ とする．$k_1 \le k_2 \le k_3$ としてよい．

(k_1, k_2, k_3) を最大公約数で割ったものを (k_1', k_2', k_3') とする．

P_1, P_2, P_3 をそれぞれ x_1, x_2, x_3 で表すことにする．

G の判別式を x_1, x_2, x_3 で表した多項式を $F_G(x_1, x_2, x_3)$ と書くことにする．

A_3, B_3, H_3 の場合，G-不変式 x_1, x_2, x_3 を上手にとれば，

$$F_{W(A_3)}, F_{W(B_3)}, F_{W(H_3)}$$

はそれぞれ定理 2.3 にある

$$F_{A,1}, F_{B,1}, F_{H,1}$$

になる．No.25, No.26 の場合にも G-不変式 x_1, x_2, x_3 を上手にとれば，それぞれ $F_{A,1}, F_{B,1}$ になる．

	group	位数	k_1, k_2, k_3	鏡映数	次数	(k'_1, k'_2, k'_3)
A_3	$W(A_3)$	24	$2, 3, 4$	2^6	12	$(2, 3, 4)$
B_3	$W(B_3)$	48	$2, 4, 6$	2^9	18	$(1, 2, 3)$
H_3	$W(H_3)$	120	$2, 6, 10$	2^{15}	30	$(1, 3, 5)$
No.24	G_{336}	336	$4, 6, 14$	2^{21}	42	$(2, 3, 7)$
No.25	G_{648}	648	$6, 9, 12$	3^{24}	36	$(2, 3, 4)$
No.26	G_{1296}	1296	$6, 12, 18$	$2^9 3^{24}$	54	$(1, 2, 3)$
No.27	G_{2160}	2160	$6, 12, 30$	2^{45}	90	$(1, 2, 5)$

The discriminant of the group G_{120}, the Coxeter group of type H_3

The discriminant of the polynominal $P(t)$ defined by

$$P(t) = t^6 + y_1 t^5 + y_2 t^3 + y_3 t + \frac{1}{20} y_2^2 - \frac{1}{4} y_1 y_3$$

is g_0^2 up to a constant factor, where

$$\begin{aligned} g_0 &= 125 y_1^3 y_2^4 + 864 y_2^5 - 1250 y_1^4 y_2^2 y_3 \\ &\quad -9000 y_1 y_2^3 y_3 + 3125 y_1^5 y_3^2 + 25000 y_1^2 y_2 y_3^2 + 50000 y_3^3 \end{aligned}$$

The polynomial g_0 is regarded as the discriminant of the group G_{120}.

$(y_1, y_2, y_3) \Longleftarrow (x_1, x_2, x_3)$

$$\begin{cases} y_1 &= -4x_1 \\ y_2 &= 10x_1^3 - 25x_2 \\ y_3 &= -4x_1^5 + 50x_1^2 x_2 - 50x_3 \end{cases}$$

Then g_0 coincides with the determinant of the matrix M up to a constant factor, where M is defined by

$$M = \begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ 3x_2 & 2x_3 + 2x_1^2 x_2 & 7x_1 x_2^2 + 2x_1^4 x_2 \\ 5x_3 & 7x_1 x_2^2 + 2x_1^4 x_2 & \frac{1}{2}(15x_2^3 + 4x_1^4 x_3 + 18x_1^3 x_2^2) \end{pmatrix}$$

We note that

$$P(-x_1) = \frac{125}{4} x_2^2$$

The hypersurface defined as the zero set of the polynomial

$$f_0 = \det M$$

is an example of logarithmic free divisors.

Define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then we have

$[V_0, V_1] = 2V_1, [V_0, V_2] = 4V_2, [V_1, V_2] = (4x_1^3x_2+2x_2^2)V_0+4x_1x_2V_1$

and $V_j f_0/f_0$ is a polynomial of x_1, x_2, x_3 $(j = 0, 1, 2)$.

Consider the system of differential equations

$$\begin{cases} V_0 u = -2u \\ V_1 u = 0 \\ \{V_2^2 + 4x_1^2(3x_2^2 + 2x_1x_3)\}u = 0 \end{cases}$$

This system is equivalent to the following one;

$$\begin{cases} V_0 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \\ V_1 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -4x_2(2x_1^3 + x_2) & 0 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \\ V_2 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -4x_1^2(3x_2^2 + 2x_1x_3) & 0 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \end{cases}$$

By taking a cycle or a path C appropriately, the function of (x_1, x_2, x_3) defined by the definite integral

$$\int_C P(t)^{-1/2} dt$$

is a solution of the system above.

This argument is generalized as follows.

Define A_j $(j = 1, 2, 3)$ by

```
aa1={{s0,0},{0,4+s0}}
aa2={{(2/15)*(2+s0)*x1^2,0},
     {(2/15)*((8+34*s0)*x1^3*x2
      +15*s0*x2^2+10*(s0+2)*x1*x3),
      (2/15)*(2+s0)*x1^2}}
aa3={{0,1},
    {(-(2/225))*(8*(s0+2)^2*x1^8
     +10*(4*s0-7)*(s0+2)*x1^5*x2
     +25*(2*s0-5)*(s0-4)*x1^2*x2^2
      -300(1+2*s0)*x1^3*x3-375*(s0+2)*x2*x3),
     (4/15)*(2+s0)*x1*(2*x1^3+5*x2)}}
```

and consider the system of differential equations

$$V_i \begin{pmatrix} u \\ V_2 u \end{pmatrix} = A_{i+1} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \quad (i = 0, 1, 2)$$

Then, putting $s_0 = -2$, we obtain the previous system.

We also have the following result. Consider the system of differential equations

$$V_i \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = B_{i+1} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (i = 0, 1, 2)$$

where B_j are defined as follows:

```
bb1={{s0, 0, 0}, {0, 2+s0, 0}, {0, 0, 4+s0}};
bb2={{0, 1, 0},
     {-2/225*x1*(8*x1^3+70*r2*x1^3-100*r2^2*x1^3
      +8*s0*x1^3+35*r2*s0*x1^3+2*s0^2*x1^3-180*x2
      +825*r2*x2-750*r2^2*x2-90*s0*x2+75*r2*s0*x2),
      1/15*(8+5*r2+4*s0)*x1^2, r2},
     {1/900*(-128*x1^6+80*r2*x1^6+100*r2^2*x1^6-128*s0*x1^6
      +40*r2*s0*x1^6-32*s0^2*x1^6-320*x1^3*x2-4000*r2*x1^3*x2
      +5500*r2^2*x1^3*x2+3280*s0*x1^3*x2-200*r2*s0*x1^3*x2-
      80*s0^2*x1^3*x2-4500*r2*x2^2+5625*r2^2*x2^2+1800*s0*x2^2
      +2400*x1*x3-3000*r2*x1*x3+1200*s0*x1*x3),
      1/15*x1*(8*x1^3+5*r2*x1^3+4*s0*x1^3+80*x2+50*r2*x2+10*s0*x2),
     1/15*(4-5*r2+2*s0)*x1^2}};
bb3={{0, 0, 1},
     {1/900*(-128*x1^6+80*r2*x1^6+100*r2^2*x1^6-128*s0*x1^6
      +40*r2*s0*x1^6-32*s0^2*x1^6-320*x1^3*x2-4000*r2*x1^3*x2
      +5500*r2^2*x1^3*x2-320*s0*x1^3*x2-200*r2*s0*x1^3*x2
      -80*s0^2*x1^3*x2-4500*r2*x2^2+5625*r2^2*x2^2+2400*x1*x3-
      3000*r2*x1*x3+1200*s0*x1*x3),
      1/15*x1*(8*x1^3+5*r2*x1^3+4*s0*x1^3+20*x2+50*r2*x2
      +10*s0*x2), 1/15*(4-5*r2+2*s0)*x1^2},
     {1/450*(-128*x1^8+80*r2*x1^8+100*r2^2*x1^8-128*s0*x1^8
      +40*r2*s0*x1^8-32*s0^2*x1^8+80*x1^5*x2-500*r2*x1^5*x2
      +500*r2^2*x1^5*x2-280*s0*x1^5*x2+200*r2*s0*x1^5*x2-
     160*s0^2*x1^5*x2-2600*x1^2*x2^2-3250*r2*x1^2*x2^2
     +8125*r2^2*x1^2*x2^2+1000*s0*x1^2*x2^2+
     625*r2*s0*x1^2*x2^2-200*s0^2*x1^2*x2^2+1200*x1^3*x3
     -1500*r2*x1^3*x3+2400*s0*x1^3*x3+3000*x2*x3
     -3750*r2*x2*x3+1500*s0*x2*x3),
     1/4*(4+5*r2)*x2*(4*x1^3+5*x2),
     1/15*x1*(16*x1^3-5*r2*x1^3+8*s0*x1^3+40*x2-50*r2*x2+20*s0*x2)}};
```

s_0, r_2 : parameters

May put $s_0 = -2$

If $r_2 = 0$, then $V_j\vec{u} = B_{j+1}\vec{u}(j = 0, 1, 2)$ has a quotient $V_j\vec{u} = A_{j+1}\vec{u}(j = 0, 1, 2)$. On the other hand, in the case $s_0 = \frac{1}{2}$, $r_2 = 1$, the monodromy group of the system $V_j\vec{u} = B_{j+1}\vec{u}\,(j = 0, 1, 2)$ coincides with $W(H_3)$.

The discriminant of the group G_{336}, ST notation No.24

$$\begin{aligned} P(t) &= t^7-\tfrac{7}{2}(c_1-1)x_2t^5-\tfrac{7}{2}(c_1-1)x_3t^4-7(c_1+4)x_2^2t^3 \\ &\quad -14(c_1+2)x_2x_3t^2+\tfrac{7}{2}\{(3c_1-7)x_2^3-(c_1+5)x_3^2\}t \\ &\quad +\tfrac{1}{2}(7c_1-131)x_2^2x_3+x_7 \\ &\quad (c_1^2=-7) \end{aligned}$$

The discriminant of $P(t)$ is f_0^2 up to a constant factor.

$$\begin{aligned} f_0 &= 2048x_2^9x_3-22016x_2^6x_3^3+60032x_2^3x_3^5-1728x_3^7 \\ &\quad +256x_2^7x_7-1088x_2^4x_3^2x_7-1008x_2x_3^4x_7 \\ &\quad +88x_2^2x_3x_7^2-x_7^3 \end{aligned}$$

f_0 is the discriminant of the unitary reflection group G_{336}.

Define vector fields V_0, V_1, V_2 by

$$^t(V_0,V_1,V_2)=M\,^t(\partial_{x_2},\partial_{x_3},\partial_{x_7})$$

Then V_0, V_1, V_2 form the generators of logarithmic vector fields along $f_0=0$. Here

$$M = \begin{pmatrix} 2x_2 & 3x_3 & 7x_7 \\ x_3^2 & -\frac{1}{12}x_7 & -\frac{4}{3}x_2(28x_2^3x_3 - 128x_3^3 + 3x_2x_7) \\ 7x_7 & -56x_2(2x_2^3 - 13x_3^2) & 28(32x_2^6 - 40x_2^3x_3^2 - 84x_3^4 + 59x_2x_3x_7) \end{pmatrix}$$

Put

```
A0={{s0, 0, 0}, {0, s0 + 4, 0}, {0, 0, s0 + 5}};

A1={{0, 1, 0},
   {1/162*x2*(-32*x2^3 + 28*c4*x2^3 + 4*c4^2*x2^3
    - 40*s0*x2^3 + 4*c4*s0*x2^3 - 8*s0^2*x2^3
    - 72*x3^2 - 129*c4*x3^2 - 15*c4^2*x3^2
    - 72*s0*x3^2 - 57*c4*s0*x3^2),
    1/9*(-10 + c4 - 4*s0)*x2^2, c4*x3/504},
   {-7/54*(-1216*x2^3*x3 + 296*c4*x2^3*x3
    + 56*c4^2*x2^3*x3 - 1376*s0*x2^3*x3 -
     112*c4*s0*x2^3*x3 - 304*s0^2*x2^3*x3
    - 144*c4*x3^3 - 18*c4^2*x3^3 - 1368*s0*x3^3 +
     24*x2*x7 + 3*c4*x2*x7 + 114*s0*x2*x7),
    -14/3*(-20 + 5*c4 - 38*s0)*x2*x3,
    -1/9*(8 + c4 + 2*s0)*x2^2}}

A2={{0, 0, 1},
   {-7/54*(-1216*x2^3*x3 + 296*c4*x2^3*x3
    + 56*c4^2*x2^3*x3 - 1520*s0*x2^3*x3 -
     112*c4*s0*x2^3*x3 - 304*s0^2*x2^3*x3
    - 144*c4*x3^3 - 18*c4^2*x3^3 + 24*x2*x7
    + 3*c4*x2*x7 + 24*s0*x2*x7),
    -14/3*(-152 + 5*c4 - 38*s0)*x2*x3,
    -1/9*(2 + c4 + 2*s0)*x2^2},
   {98/9*(-1152*x2^5 + 240*c4*x2^5 + 48*c4^2*x2^5
    - 1728*s0*x2^5 - 48*c4*s0*x2^5 - 1760*x2^2*x3^2
    + 388*c4*x2^2*x3^2 + 76*c4^2*x2^2*x3^2
    - 2632*s0*x2^2*x3^2 - 356*c4*s0*x2^2*x3^2
    - 2888*s0^2*x2^2*x3^2 + 24*x3*x7 + 3*c4*x3*x7
    + 114*s0*x3*x7),
    -1176*(-2 + c4)*(2*x2^3 - x3^2),
    14/3*(190 + 5*c4 + 76*s0)*x2*x3}}
```

There is a system of differential equation of rank three defined by

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2)$$

This system has two parameters s_0, c_4.

Substituting $s_0 = -1, c_4 = 0$ in A_j, we obtain $A_j^{(0)}$;

$$A_0^{(0)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

$$A_1^{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}x_2^2 & 0 \\ \frac{7}{3}(8x_2^3 x_3 - 76x_3^3 + 5x_2 x_7) & -84x_2 x_3 & -\frac{2}{3}x_2^2 \end{pmatrix},$$

$$A_2^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 532x_2 x_3 & 0 \\ 196(32x_2^5 - 112x_2^2 x_3^2 - 5x_3 x_7) & 2352(2x_2^3 - x_3^2) & 532x_2 x_3 \end{pmatrix}$$

The system

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j^{(0)} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2)$$

has a quotient which is defined by $V_1 u = 0$.

We now study the restriction of the system

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j^{(0)} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2), \quad V_1 u = 0$$

to the hyperplane $x_2 = 0$. Then we obtain an ordinary differential equation

$$\left(\partial_{x_7}^2 + \frac{18x_7^2}{7(1728x_3^7 + x_7^3)} \partial_{x_7} + \frac{10x_7}{49(1728x_3^7 + x_7^3)} \right) u = 0$$

One of its solutions is

$$x_3^{-1/3} F\left(\frac{1}{21}, \frac{10}{21}; \frac{2}{3}; -\frac{x_7^3}{1728x_3^7} \right)$$

Similarly as the restriction to $x_3 = 0$, we obtain an ordinary differential equation

$$\left(\partial_{x_7}^2 - \frac{256x_2^7 + 11x_7^2}{7x_7(256x_2^7 - x_7^2)} \partial_{x_7} + \frac{3}{49(-256x_2^7 + x_7^2)} \right) u = 0$$

One of its solutions is

$$x_2^{-1/2} F\left(\frac{1}{14}, \frac{3}{14}; \frac{3}{7}; \frac{x_7^2}{256x_2^7} \right)$$

Last we note that, in the case $c_4 = -9$, $s_0 = 1/2$, the system of differential equations has a finite monodromy group.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
kn1 = (x^6*y^3)/32 + (3*x^3*y^5)/28 + (3*y^7)/49
     - (3/16)*x^4*y^2*z - (3/7)*x*y^4*z + z^3;
kn2 = (-(1/864))*x^6*y^3 + (5*x^3*y^5)/84
     + (3*y^7)/49 - (1/48)*x^4*y^2*z - (3/7)*x*y^4*z
     + z^3;
kn3 = (x^3*y^5)/21 + (3*y^7)/49 - (3/7)*x*y^4*z + z^3;
kn4 = (3*y^7)/49 - (3/7)*x*y^4*z + z^3;
kn5 = (78125*x^9*y)/200120949 + (44375*x^6*y^3)/4840416
     + (107*x^3*y^5)/1372 + (3*y^7)/49 -
     (6250*x^7*z)/3176523 - (1375*x^4*y^2*z)/16464
     - (3/7)*x*y^4*z + z^3;
kn6 = (64*x^9*y)/823543 + (208*x^6*y^3)/453789
     + (68*x^3*y^5)/1029 + (3*y^7)/49 + (48*x^7*z)/117649
     - (40*x^4*y^2*z)/1029 - (3/7)*x*y^4*z + z^3;
kn7 = -((448*x^9*y)/243) + (16*x^6*y^3)/9 - (4*x^3*y^5)/7
     + (3*y^7)/49 - (112*x^7*z)/27 +
     (8/3)*x^4*y^2*z - (3/7)*x*y^4*z + z^3;
kn8 = -((752*x^9*y)/823543) - (2017*I*x^9*y)/(823543*Sqrt[3])
     - (397*x^6*y^3)/33614 + (323*I*Sqrt[3]*x^6*y^3)/33614
     + (39*x^3*y^5)/686 + (9/686)*I*Sqrt[3]*x^3*y^5
     + (3*y^7)/49 + (1763*x^7*z)/235298
     - (249*I*Sqrt[3]*x^7*z)/235298 + (3/686)*x^4*y^2*z -
     (37/686)*I*Sqrt[3]*x^4*y^2*z - (3/7)*x*y^4*z + z^3;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Th discriminant of the unitary reflection group G_{2160}, ST notation No.27

file "c080903a", "c090101f"

Consider the polynomial

$$P(t) = t^6 + y_1 t^5 + y_2 t^4 + y_3 t^3 + y_4 t^2 + y_5 t + y_6$$

Substitute y_j $(j = 1, 2, \ldots, 6)$ by x_j $(j = 1, 2, \ldots, 6)$;

```
y1=x1
y2=(5/16)*(9 + sr)*x2,
y3=(5/64)*(11 + 3*sr)*x1*x2,
y4=(5/512)*(37 + 45*sr)*x2^2,
y5=(61 + 5*sr)*(-64*x1^3*x2 + 373*x1*x2^2
           + 15*sr*x1*x2^2 + 2*x5))/12288,
y6=(-279 + 145*sr)*(-512*x1^4*x2 + 2864*x1^2*x2^2
                    + 1425*x2^3 + 135*sr*x2^3 +
                    16*x1*x5)/3538944,
```

where $sr^2 = -15$. Then the discriminant of the polynomial $P(t)$ is f_0^2, where

$$\begin{aligned} f_0 \;=\;& 65536x_1^{11}x_2^2 - 1765376x_1^9x_2^3 + 17406016x_1^7x_2^4 \\ & -73887360x_1^5x_2^5 + 107371008x_1^3x_2^6 \\ & +34338816x_1x_2^7 - 4096x_1^8x_2x_5 + 96640x_1^6x_2^2x_5 \\ & -707952x_1^4x_2^3x_5 + 1622592x_1^2x_2^4x_5 \\ & +186624x_2^5x_5 + 64x_1^5x_5^2 - 1584x_1^3x_2x_5^2 \\ & +7128x_1x_2^2x_5^2 + 9x_5^3 \end{aligned}$$

up to a constant factor.

The polynomial f_0 is regarded as the discriminant of the unitary reflection group No.27. In particular, f_0 is obtained as the determinant of the matrix

$$M = \begin{pmatrix} x_1 & 2x_2 & 5x_5 \\ x_2^2 & \frac{1}{432}(144x_1x_2^2 - x_5) & \frac{1}{108}(640x_1^6x_2 - 9388x_1^4x_2^2 + 36600x_1^2x_2^3 \\ & & -19872x_2^4 - 28x_1^3x_5 + 307x_1x_2x_5) \\ x_5 & \frac{1}{135}(-1920x_1^4x_2 + 8724x_1^2x_2^2 & -\frac{4}{135}x_1(65920x_1^6x_2 - 887092x_1^4x_2^2 + 2886120x_1^2x_2^3 \\ & +16416x_2^3 + 139x_1x_5) & +367632x_2^4 - 2692x_1^3x_5 + 20533x_1x_2x_5) \end{pmatrix}$$

and is a discriminant of G_{2160}.

We define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then V_0, V_1, V_2 form generators of the logarithmic vector fields along the set $f_0 = 0$ in the (x_1, x_2, x_5)-space. By direct computation, we have

$$\begin{aligned} & [V_1, V_2] \\ = \; & \tfrac{1}{540}(3200x_1^5x_2 - 16412x_1^3x_2^2 - 18056x_1x_2^3 \\ & \qquad -80x_1^2x_3 - 307x_2x_3)V_0 \\ & -\tfrac{8}{135}(474x_1^4 - 4102x_1^2x_2 + 7209x_2^2)V_1 \\ & -\tfrac{1}{54}x_1(6x_1^2 - 73x_2)V_2 \end{aligned}$$

We consider the system of differential equations

$$V_j \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} = A_{j+1} \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} \quad (j = 0, 1, 2)$$

where A_1, A_2, A_3 are matrices of rank three defined as follows.

```
A1={{(1/264)*(1344 + 5*h2 + 34560*h1), 0, 0},
    {0, (1/264)*(2136 + 5*h2 + 34560*h1), 0},
    {0, 0, (5/264)*(480 + h2 + 6912*h1)}}

A2={{0, 1, 0},
    {(1/36582036480)*(-873538560*x1^6 - 5462400*h2*x1^6
      - 8000*h2^2*x1^6 - 53084160000*h1*x1^6 -
      293068800*h2*h1*x1^6 + 6680960040960*h1^2*x1^6
      + 23319480960*x1^4*x2 + 144298800*h2*x1^4*x2
      + 210000*h2^2*x1^4*x2 + 2493684817920*h1*x1^4*x2 +
      10156492800*h2*h1*x1^4*x2 - 45967909847040*h1^2*x1^4*x2
      - 148925977920*x1^2*x2^2 - 929415600*h2*x1^2*x2^2
      - 1378125*h2^2*x1^2*x2^2 - 12594093705216*h1*x1^2*x2^2 -
      48930635520*h2*h1*x1^2*x2^2 + 153739955404800*h1^2*x1^2*x2^2
      + 59338033920*x2^3 + 224532000*h2*x2^3
      - 10329880264704*h1*x2^3 - 36582036480*h2*h1*x2^3 -
      758565108449280*h1^2*x2^3 - 137356560*x1*x3
      - 519750*h2*x1*x3 - 2915066880*h1*x1*x3),
     -((1/213840)*(x1*(68280*x1^2 + 200*h2*x1^2
     + 12673152*h1*x1^2 - 907560*x2 -
      2625*h2*x2 - 84177792*h1*x2))),
     (-(1/6))*h1*(x1^2 - 6*x2)},
   {(1/11431886400)*(69009546240*x1^7 + 431529600*h2*x1^7
     + 632000*h2^2*x1^7 + 4193648640000*h1*x1^7
     + 23152435200*h2*h1*x1^7 - 527795843235840*h1^2*x1^7 -
      1387644704640*x1^5*x2 - 8779309200*h2*x1^5*x2
     - 13170000*h2^2*x1^5*x2 - 199060235182080*h1*x1^5*x2
     - 824882227200*h2*h1*x1^5*x2 + 3125118432706560*h1^2*x1^5*x2 +
      7236937923840*x1^3*x2^2 + 45847706400*h2*x1^3*x2^2
     +69330375*h2^2*x1^3*x2^2 + 858016346793984*h1*x1^3*x2^2
     + 3461777844480*h2*h1*x1^3*x2^2
     - 10904755042713600*h1^2*x1^3*x2^2 - 7612774139520*x1*x2^3
     - 47225494800*h2*x1*x2^3 - 70166250*h2^2*x1*x2^3
     + 872986623860736*h1*x1*x2^3 + 3045177400320*h2*h1*x1*x2^3 +
      5803023079636992 0*h1^2*x1*x2^3 + 6371220240*x1^2*x3
     + 24131250*h2*x1^2*x3 + 133600389120*h1*x1^2*x3
     - 14543899920*x2*x3 - 52925400*h2*x2*x3
     - 457275456000*h1*x2*x3),
      (1/133650)*(2995680*x1^4 + 15800*h2*x1^4
     + 2253540096*h1*x1^4 - 19577160*x1^2*x2 -
```

```
    121875*h2*x1^2*x2 - 12962509056*h1*x1^2*x2
    + 133650*h2*x2^2),
   -((1/427680)*(x1*(98640*x1^2 + 200*h2*x1^2 - 21199104*h1*x1^2
   - 1271880*x2 - 2625*h2*x2 + 113923584*h1*x2)))}}

A3={{0, 0, 1},
   {(1/11431886400)*(69009546240*x1^7 + 431529600*h2*x1^7
   + 632000*h2^2*x1^7 + 4193648640000*h1*x1^7
   + 23152435200*h2*h1*x1^7 - 527795843235840*h1^2*x1^7 -
     1732525856640*x1^5*x2 - 10062349200*h2*x1^5*x2
   - 13170000*h2^2*x1^5*x2 - 207928607662080*h1*x1^5*x2
   - 824882227200*h2*h1*x1^5*x2 + 3125118432706560*h1^2*x1^5*x2 +
     9005747132160*x1^3*x2^2 + 52428097800*h2*x1^3*x2^2
   +69330375*h2^2*x1^3*x2^2 + 903500012150784*h1*x1^3*x2^2
   + 3461777844480*h2*h1*x1^3*x2^2
   - 10904755042713600*h1^2*x1^3*x2^2 - 5666782239360*x1*x2^3
   - 39985941600*h2*x1*x2^3 - 70166250*h2^2*x1*x2^3
   + 923026415579136*h1*x1*x2^3 + 3045177400320*h2*h1*x1*x2^3 +
     5803023079636992O*h1^2*x1*x2^3 + 14993249040*x1^2*x3
   + 56207250*h2*x1^2*x3 + 355309701120*h1*x1^2*x3
   + 18543135600*x2*x3 + 70166250*h2*x2*x3
   + 393534028800*h1*x2*x3),
   (1/133650)*(6749760*x1^4 + 15800*h2*x1^4
   + 2253540096*h1*x1^4 - 52065000*x1^2*x2 -
     121875*h2*x1^2*x2 - 12962509056*h1*x1^2*x2
   + 57095280*x2^2 + 133650*h2*x2^2),
   -((1/427680)*(x1*(51120*x1^2 + 200*h2*x1^2 - 21199104*h1*x1^2
   - 693720*x2 - 2625*h2*x2 + 113923584*h1*x2)))},
  {(1/3572464500)*(-5451754152960*x1^8 - 34090838400*h2*x1^8
   - 49928000*h2^2*x1^8 -
     331298242560000*h1*x1^8 - 1829042380800*h2*h1*x1^8
   + 41695871615631360*h1^2*x1^8 + 92679446021760*x1^6*x2
   + 557129242800*h2*x1^6*x2 + 770250000*h2^2*x1^6*x2 +
     16376190696529920*h1*x1^6*x2 + 66944720332800*h2*h1*x1^6*x2
   - 206882987012259840*h1^2*x1^6*x2 - 486812026399680*x1^4*x2^2
   - 2851829440200*h2*x1^4*x2^2 - 3815371125*h2^2*x1^4*x2^2
   - 65786091386351616*h1*x1^4*x2^2
   - 258562921547520*h2*h1*x1^4*x2^2
   + 720172257804288000*h1^2*x1^4*x2^2 +
     794826767275200*x1^2*x2^3 + 4693483812600*h2*x1^2*x2^3
```

```
+ 6515437500*h2^2*x1^2*x2^3 - 85148625690906624*h1*x1^2*x2^3
- 322688758106880*h2*h1*x1^2*x2^3 -
  4784402313213050880*h1^2*x1^2*x2^3 - 228079851943680*x2^4
- 1806238051200*h2*x2^4 - 3572464500*h2^2*x2^4
- 18134447123865600*h1*x2^4 - 74078623872000*h2*h1*x2^4 -
  571889918160*x1^3*x3 - 2140701750*h2*x1^3*x3
- 13779685370880*h1*x1^3*x3 + 1951711692480*x1*x2*x3
+ 7085989350*h2*x1*x2*x3 + 62513711884800*h1*x1*x2*x3),
-((1/10125)*(16*(-1248200*x1^5 + 2998076544*h1*x1^5
+ 8604480*x1^3*x2 - 16280694144*h1*x1^3*x2 - 7239780*x1*x2^2
- 3212265600*h1*x1*x2^2 + 180225*x3))),
(1/66825)*(5915520*x1^4 + 15800*h2*x1^4 - 962955648*h1*x1^4
- 45630000*x1^2*x2 - 121875*h2*x1^2*x2 + 5217654528*h1*x1^2*x2
+ 50038560*x2^2 + 133650*h2*x2^2 + 1385683200*h1*x2^2)}}}
```

A_1, A_2, A_3 contain parameters h_1, h_2. The determination of A_1, A_2, A_3 was accomplished by Masayuki Noro (Kobe Univ.).

The case $h_1 = -\frac{19}{5184}, h_2 = -\frac{704}{3}$

In this case the monodromy group of the system of differential equations becomes G_{2160}.

The case $h_1 = 0, h_2 = -\frac{2136}{5}$

In this case there is a quotient of the system above. In fact,

$$\left\{ \begin{array}{ccc} V_1 u & = & \frac{1}{162} x_1 (13x_1^2 - 162x_2) u \\ V_j \begin{pmatrix} u \\ V_2 u \end{pmatrix} & = & B_{j+1} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2) \end{array} \right.$$

is a quotient of the system $V_j \vec{u} = A_{j+1} \vec{u}$ $(j = 0, 1, 2)$ defined above, where B_j $(j = 1, 2, 3)$ are matrices of rank two defined below :

```
B1={{-3, 0}, {0, 1}};
B2={{(1/162)*x1*(13*x1^2 - 162*x2), 0},
   {(-98592*x1^7 + 1926304*x1^5*x2
    - 10970316*x1^3*x2^2 + 17754552*x1*x2^3
    - 15066*x1^2*x3 + 30861*x2*x3)/43740,
   (-1350*x1^3 + 15390*x1*x2)/43740}};
B3={{0, 1},
    {-((1/164025)*(4*(-6490640*x1^8
     + 180214176*x1^6*x2 - 999084132*x1^4*x2^2 +
    712058040*x1^2*x2^3 + 1244595456*x2^4
    - 2995542*x1^3*x3 + 665577*x1*x2*x3))),
    -((4*(511920*x1^4 - 3948750*x1^2*x2
    + 4330260*x2^2))/164025)}};
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
ks = z*(s*x^2*y^4 + 2*y^5 - 10*x*y^2*z + 5*z^2)

gk01 = z*(25*x^4*y^3 - 20*x^2*y^4 + 4*y^5 + 50*x^3*y*z
      - 20*x*y^2*z + 10*z^2)
gk02 = z*(25*x^4*y^3 + 80*x^2*y^4 + 64*y^5 - 320*x*y^2*z
      + 160*z^2)
gk03 = (-(1125/4))*x^7*y^4 - (225*x^5*y^5)/2 + 25*x^3*y^6
      - (4*x*y^7)/3 - (675/2)*x^4*y^3*z + 45*x^2*y^4*z
      - 2*y^5*z - (225/2)*x^3*y*z^2 - 5*z^3
gk04 = z*(600*x^4*y^3 - 1080*x^2*y^4 + 486*y^5 - 1000*x^5*z
      + 3600*x^3*y*z - 2430*x*y^2*z + 1215*z^2)
gk05 = z*(-45*x^2*y^4 + 54*y^5 + 200*x^3*y*z - 270*x*y^2*z
      + 135*z^2)
gk06 = 4*x*y^7 + 6*y^5*z + 15*z^3
gk07 = -50*x^5*y^5 - 384*x*y^7 + 75*x^4*y^3*z
      + 720*x^2*y^4*z - 576*y^5*z - 1440*z^3
gk08 = z*(75*x^4*y^3 - 540*x^2*y^4 + 972*y^5 - 500*x^5*z
      + 3150*x^3*y*z - 4860*x*y^2*z + 2430*z^2)
gk09 = z*(-375*x^6*y^2 + 1350*x^4*y^3 - 1620*x^2*y^4
      + 648*y^5 - 2000*x^5*z + 5100*x^3*y*z - 3240*x*y^2*z
      + 1620*z^2)
gk10 = (-4096*
        x^7*y^4 + 1792*x^5*y^5 - 200*x^3*y^6 - 6400*x^4*y^3*z
      + 2500*x^2*y^4*z - 250*y^5*z + 5120*x^5*z^2
      - 8000*x^3*y*z^2 + 1250*x*y^2*z^2 - 625*z^3)
gk11 = (-3750*x^7*y^4 - 362925*x^5*y^5 + 21018069*x^3*y^6
      + 1701000*x*y^7 - 36009450*x^2*y^4*z - 1350000*y^5*z
      + 62500*x^3*y*z^2 + 19507500*x*y^2*z^2 - 3375000*z^3)
gk12 = (96*x^9*y^3 - 288*x^7*y^4 + 576*x^5*y^5 - 600*x^3*y^6
      + 250*x*y^7 - 1440*x^6*y^2*z + 1800*x^4*y^3*z
      - 750*y^5*z + 4860*x^5*z^2 - 9000*x^3*y*z^2
      + 5625*x*y^2*z^2 - 1875*z^3)
gk13 = (62208*x^5*y^5 - 183775*x^3*y^6 + 144000*x*y^7
      - 1440000*x^4*y^3*z + 4275000*x^2*y^4*z - 3456000*y^5*z
      + 6635520*x^5*z^2 - 20736000*x^3*y*z^2
      + 18360000*x*y^2*z^2 - 8640000*z^3)

gk14,....,gk35
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The discriminant of the group $W(B_3)$ and the unitary reflection group G_{1296}, ST notation No.26

In this case we start with the matrix

$$M = \begin{pmatrix} x_1 & 2x_2 & 3x_3 \\ 2x_2 & x_1x_2 + 3x_3 & 2x_1x_3 \\ 3x_3 & 2x_1x_3 & x_2x_3 \end{pmatrix}$$

The determinant of M coincides with f_0 up to signature;

$$f_0 = x_3(-x_1^2x_2^2 + 4x_2^3 + 4x_1^3x_3 - 18x_1x_2x_3 + 27x_3^2)$$

Let V_0, V_1, V_2 be vector fields defined by

$${}^t(V_0, V_1, V_2) = M^t(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$$

We define

$$
\begin{aligned}
A_1 &= \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 1+s_0 & 0 \\ 0 & 0 & 2+s_0 \end{pmatrix} \\
A_2 &= \begin{pmatrix} 0 & 1 & 0 \\ -(-1+2k_1-s_0)s_0x_2 & k_1x_1 & 1+s_0 \\ -(-1+3k_1-2s_0)s_0x_3 & 0 & (1+k_1)x_1 \end{pmatrix} \\
A_3 &= \begin{pmatrix} 0 & 0 & 1 \\ -(-2+3k_1-2s_0)s_0x_3 & 0 & k_1x_1 \\ 0 & (1-3k_1+2s_0)x_3 & (2k_1-s_0)x_2 \end{pmatrix}
\end{aligned}
$$

and consider the system of differential equations

$$V_j\vec{u} = A_{j+1}\vec{u} \quad (j = 0, 1, 2)$$

where

$$\vec{u} = {}^t(u, V_1u, V_2u)$$

In the case $W(B_3)$, by taking a linear coordinate $t = (t_1, t_2, t_3)$ of the standard representation space, we have

$$\begin{aligned} x_1 &= t_1^2 + t_2^2 + t_3^2 \\ x_2 &= t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2 \\ x_3 &= t_1^2 t_2^2 t_3^2 \end{aligned}$$

It is provable that in the case $s_0 = k_1 = \frac{1}{2}$, the system of differential equations

$$V_j \vec{u} = A_{j+1} \vec{u} \quad (j = 0, 1, 2)$$

on x-space has solutions t_1, t_2, t_3, regarding them as functions of x_1, x_2, x_3.

On the other hand, in the case of the unitary reflection group No.26, by taking a linear coordinate $t = (t_1, t_2, t_3)$ of the standard representation space appropriately, we may take y_1, y_2, y_3 as basic invariants, where

$$\begin{aligned} y_1 &= t_1^6 - 10t_1^3t_2^3 + t_2^6 - 10t_1^3t_3^3 - 10t_2^3t_3^3 + t_3^6 \\ y_2 &= (t_1^3 + t_2^3 + t_3^3)((t_1^3 + t_2^3 + t_3^3)^3 + 216(t_1t_2t_3)^3) \\ y_3 &= ((t_1^3 - t_2^3)(t_2^3 - t_3^3)(t_3^3 - t_1^3))^2 \end{aligned}$$

If we define

$$x_1 = y_1,\ x_2 = (y_1^2 - y_2)/3,\ x_3 = 16y_3,$$

the discriminant of this group coincides f_0 up to a constant fanctor. Moreover, we find that in the case $s_0 = 1/6,\ k_2 = 1/3$, the system of differential equations

$$V_j\vec{u} = A_{j+1}\vec{u} \quad (j = 0, 1, 2)$$

on x-space has solutions t_1, t_2, t_3, regarding them as functions of x_1, x_2, x_3.

We consider the restriction of the system

$$V_j\vec{u} = A_{j+1}\vec{u} \quad (j = 0, 1, 2)$$

on the hyperplane $x_1 = 0$. Then we obtain an ordinary differential equation whose solution is

$$ {}_3F_2\left(-\frac{s_0}{6}, \frac{2-s_0}{6}, \frac{4-s_0}{6}; \frac{1}{2}, \frac{s_0-2k_1+2}{2}; -\frac{27x_3^2}{4x_2^3}\right)$$

Similarly it is possible to restrict the system to the hypersurface $4x_2 - x_1^2 = 0$ and the hyperplane $x_2 = 0$. When we restrict it to the hypersurface $4x_2 - x_1^2 = 0$, we obtain an ordinary differential equation whose solution is

$$ {}_3F_2\left(\frac{2-s_0}{3}, \frac{1-s_0}{3}, -\frac{s_0}{3}; -s_0+k_1+\frac{1}{2}, s_0-2k_1+1; \frac{54x_3}{x_1^3}\right)$$

On the other hand, we restrict it to $x_2 = 0$, we obtain an ordinary differential equation whose solution is

$$ {}_3F_2\left(\frac{-s_0+2}{3}, \frac{-s_0+1}{3}, -\frac{s_0}{3}; \frac{2-k_1}{2}, \frac{3-k_1}{2}; -\frac{27x_3}{4x_1^3}\right)$$

$_3F_2(a_1, a_2, a_3; b_1, b_2; x)$

order	degree	Results in Kato	
60	2	$\left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}; \frac{2}{3}, \frac{4}{3}\right)$	$\left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}; \frac{2}{3}, \frac{4}{3}\right)$
60	6	$\left(\frac{5}{10}, \frac{7}{10}, \frac{13}{10}; \frac{7}{5}, \frac{8}{5}\right)$	$\left(\frac{7}{10}, \frac{9}{10}, \frac{13}{10}; \frac{8}{5}, \frac{9}{5}\right)$
168	4	$\left(\frac{9}{14}, \frac{11}{14}, \frac{15}{14}; \frac{4}{3}, \frac{5}{3}\right)$	$\left(\frac{9}{14}, \frac{11}{14}, \frac{15}{14}; \frac{4}{3}, \frac{5}{3}\right)$
168	6	$\left(\frac{3}{14}, \frac{5}{14}, \frac{13}{14}; \frac{3}{4}, \frac{5}{4}\right)$	$\left(\frac{13}{28}, \frac{17}{28}, \frac{33}{28}; \frac{5}{4}, \frac{6}{4}\right)$
360	6	$\left(\frac{7}{30}, \frac{13}{30}, \frac{25}{30}; \frac{3}{4}, \frac{5}{4}\right)$	$\left(\frac{7}{30}, \frac{13}{30}, \frac{25}{30}; \frac{3}{4}, \frac{5}{4}\right)$
360	12	$\left(-\frac{1}{30}, \frac{5}{30}, \frac{11}{30}; \frac{1}{5}, \frac{4}{5}\right)$	$\left(-\frac{1}{30}, \frac{5}{30}, \frac{11}{30}; \frac{1}{5}, \frac{4}{5}\right)$
216	6	$\left(\frac{2}{9}, \frac{5}{9}, \frac{8}{9}; \frac{3}{4}, \frac{5}{4}\right)$	$- - -? - --$
216	12	$\left(\frac{4}{9}, \frac{7}{9}, \frac{10}{9}; \frac{7}{6}, \frac{9}{6}\right)$	$- - -? - --$
216	12	$\left(\frac{1}{9}, \frac{4}{9}, \frac{7}{9}; \frac{3}{6}, \frac{7}{6}\right)$	$- - -? - --$