パンルヴェ方程式
と
ウェイト系

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Contents

An ODE on the complex plane.

$$\frac{dy}{dz} = f(z, y), \quad y \in \mathbf{C}^n, z \in \mathbf{C}.$$

A study of singularities of solutions.

fixed sing. $\Rightarrow$ sing of f

movable sing. $\Rightarrow$ sing depending on initial cond.

Painleve property:

ODE is said to have the Painleve property if any movable singularities are poles.

Ex. $y' = y^2 \quad \Rightarrow \quad y = -(z-c)^{-1}.$

$y' = y^3 \quad \Rightarrow \quad y = (-2z-c)^{-1/2}.$

Thm. (Poincare, Fuchs)

If a first order ODE has the Painleve property, it is equivalent to one of

(i) Solvable.

(ii) Riccati.

(iii) Weierstrass.

$$\mathrm{R}: \frac{dy}{dz} = a(z)y^2 + b(z)y + c(z).$$

$$\mathrm{W}: \left(\frac{dy}{dz}\right)^2 = 4y^3 - g_2 y - g_3.$$

Thm. (Painleve, Gambier, 1900)

If a second order ODE has the Painleve property, it is equivalent to one of

(i) Solvable.

(ii) Linear.

(iii) Weierstrass.

(iv) the Painleve equations P1 to P6.

$$\mathrm{P_I}: \ \frac{d^2y}{dz^2} = 6y^2 + z.$$

$$\mathrm{P_{II}}: \ \frac{d^2y}{dz^2} = 2y^3 + zy + \alpha.$$

$$\mathrm{P_{III}}: \ \frac{d^2y}{dz^2} = \frac{1}{y}\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$\mathrm{P_{IV}}: \frac{d^2y}{dz^2} = \frac{1}{2y}\left(\frac{dy}{dz}\right)^2 + \frac{3}{2}y^3 + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}$$

$$\mathrm{P_{V}}: \frac{d^2y}{dz^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dz}\right)^2 - \frac{1}{z}\frac{dy}{dz} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{z} + \delta\frac{y(y+1)}{y-1}$$

$$\mathrm{P_{VI}}: \frac{d^2y}{dz^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)\left(\frac{dy}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)\frac{dy}{dz} + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right)$$

Painleve equations are written as Hamiltonian systems.

$$(\mathrm{P_J}): \ \frac{dx}{dz} = -\frac{\partial H_J}{\partial y}, \quad \frac{dy}{dz} = \frac{\partial H_J}{\partial x},$$

Hamiltonian functions are given by

$$\begin{aligned}
H_{\mathrm{I}} &= \frac{1}{2}x^2 - 2y^3 - zy,\\
H_{\mathrm{II}} &= \frac{1}{2}x^2 - \frac{1}{2}y^4 - \frac{1}{2}zy^2 - \alpha y,\\
H_{\mathrm{IV}} &= -xy^2 + x^2y - 2xyz - 2\alpha x + 2\beta y,\\
zH_{\mathrm{III}} &= x^2y^2 - xy^2 + zx + (\alpha+\beta)xy - \alpha y,\\
zH_{\mathrm{V}} &= x(x+z)y(y-1) + \alpha_2 yz - \alpha_3 xy - \alpha_1 x(y-1),\\
z(z-1)H_{\mathrm{VI}} &= y(y-1)(y-z)x^2 + \alpha_2(\alpha_1+\alpha_2)(y-z)\\
&\quad - (\alpha_4(y-1)(y-z) + \alpha_3 y(y-z) + \alpha_0 y(y-1))\, x.
\end{aligned}$$

Lax equations.

$A, B:$ $n \times n$ matrix with a spectral parameter λ .

- Lax eq. in the sense of spectral preserving:

$$\frac{\partial B}{\partial z} = [A, B] \Rightarrow \text{Integrable system}$$

- Lax eq. in the sense of monodromy preserving:

$$\frac{\partial \psi}{\partial z} = A\psi, \quad \frac{\partial \psi}{\partial \lambda} = B\psi. \Rightarrow \frac{\partial B}{\partial z} = [A, B] + \frac{\partial A}{\partial \lambda}$$

$\Rightarrow$ Painleve property.
(usually, non-autonomous Hamiltonian system)

It is possible to find new Painleve eqs. by
the bi-Poisson theory on Lie algebras.

The space of initial conditions.

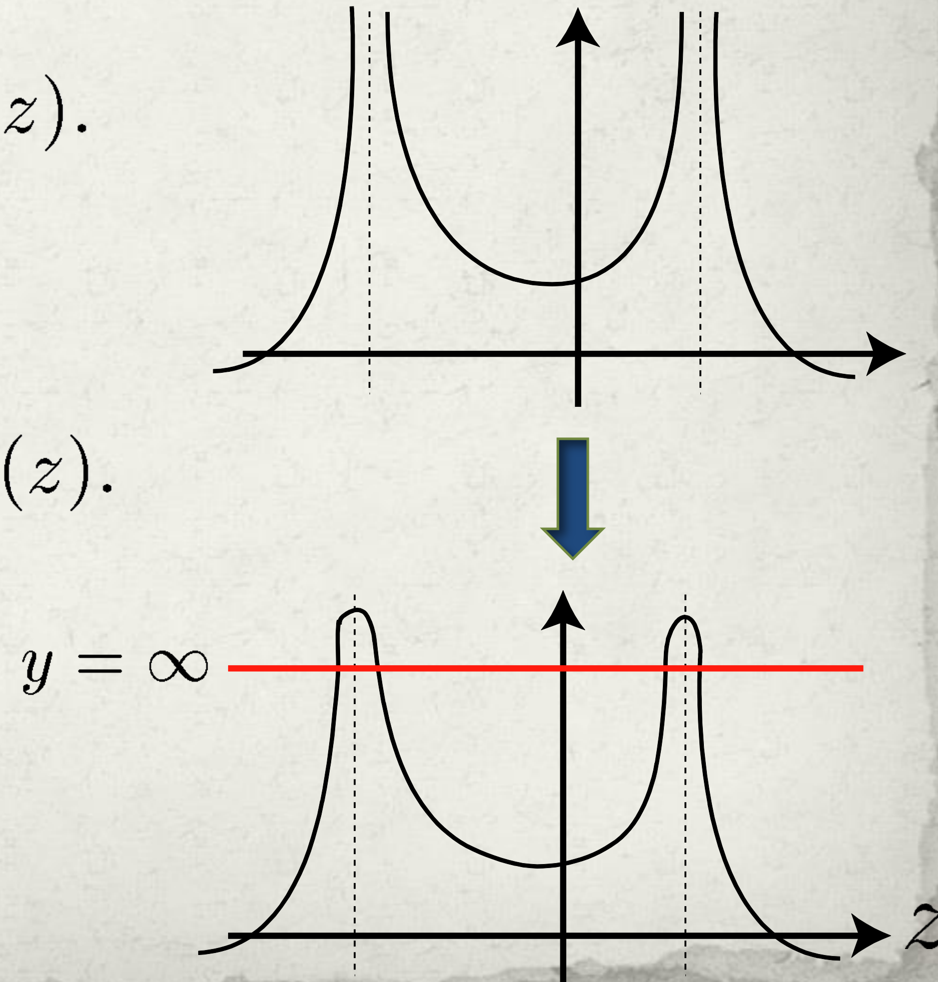
Riccati equation. Any solution is meromorphic.
(Putting $y = u'/u$, u satisfies a linear equation.)

$$\frac{dy}{dz} = a(z)y^2 + b(z)y + c(z).$$

$$y = 1/\xi$$

$$\frac{d\xi}{dz} = -c(z)\xi^2 - b(z)\xi - a(z).$$

$y : \mathbf{C} \to \mathbf{C}P^1$ is holomorphic.



$\mathbf{C}P^1$ is called the **space of initial conditions**.

The space of initial conditions …

A fiber sp. of a fiber bundle, on which any solutions have analytic continuations for any z.

1-dim:

Riccati $\longleftrightarrow$ $\mathbf{C}P^1$ $(g=0)$

Weierstrass $\longleftrightarrow$ torus $(g=1)$

Solvable $\longleftrightarrow$ $(g \geq 2)$

2-dim:

$(P_I) \sim (P_{IV})$ $\longleftrightarrow$ a certain class of alg. surfaces characterized by the nine points blowup of $\mathbf{C}P^2$ and the Dynkin diagrams D.

	$(\mathrm{P_I})$	$(\mathrm{P_{II}})$	$(\mathrm{P_{III}})$	$(\mathrm{P_{IV}})$	$(\mathrm{P_V})$	$(\mathrm{P_{VI}})$
D	E_8	E_7	D_6	E_6	D_5	D_4

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Newton diagram of ODEs.

Ex: The first Painleve equation.

$$\begin{cases} \dfrac{dx}{dz} = \overset{\cancel{(0,2,0)}\ (-1,2,1)}{6y^2} + \overset{\cancel{(0,0,1)}\ (-1,0,2)}{z} \\ \dfrac{dy}{dz} = \overset{\cancel{(1,0,0)}\ (1,-1,1)}{x}, \end{cases}$$

Newton diagram is the convex hull of these points in $\mathbf{R}^3$. In this example, they lie on the plane

$$3x + 2y + 4z = 5.$$

Newton diagram $\Longleftrightarrow$ Toric variety

In this example, the associated toric variety is the weighted projective space $\mathbf{C}P^3(3,2,4,5)$.

This space provides a suitable compactification of the natural phase space;

$$\mathbf{C}^3 = \{(x,y,z)\} \subset \mathbf{C}P^3(3,2,4,5)$$

The weighted $\mathbf{C}^*$ action:

$$(x, y, z, \varepsilon) \mapsto (\lambda^3 x, \lambda^2 y, \lambda^4 z, \lambda^5 \varepsilon). \quad \lambda \neq 0.$$

The quotient space is called the weighted projective space $\mathbf{C}_0^4/\sim = \mathbf{C}P^3(3, 2, 4, 5)$.

A weighted projective space is an orbifold (algebraic variety) with singularities.

orbifold: $M \simeq \bigcup U_\alpha / \Gamma_\alpha .$

U_α : manifold

Γ_α : finite group

$\mathbf{C}P^3(3,2,4,5)$ is defined by $[x,y,z,\varepsilon] \sim [\lambda^3 x, \lambda^2 y, \lambda^4 z, \lambda^5 \varepsilon]$.

(i) When $x \neq 0$,

$$[x,y,z,\varepsilon] \sim [1, \frac{y}{x^{2/3}}, \frac{z}{x^{4/3}}, \frac{\varepsilon}{x^{5/3}}] := [1, Y_1, Z_1, \varepsilon_1].$$

$$\sim [1, \omega Y_1, \omega^2 Z_1, \omega \varepsilon_1].$$

The subset $\{x \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_3$.

(ii) When $y \neq 0$,

$$[x,y,z,\varepsilon] \sim [\frac{x}{y^{3/2}}, 1, \frac{z}{y^2}, \frac{\varepsilon}{y^{5/2}}] := [X_2, 1, Z_2, \varepsilon_2].$$

$$\sim [-X_2, 1, Z_2, -\varepsilon_2].$$

The subset $\{y \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_2$.

(iii) The subset $\{z \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_4$.

(iv) The subset $\{\varepsilon \neq 0\}$ is homeo. to $\mathbf{C}^3/\mathbf{Z}_5$.

We obtain

$$\boxed{\mathbf{C}P^3(3,2,4,5) = \mathbf{C}^3/\mathbf{Z}_3 \cup \mathbf{C}^3/\mathbf{Z}_2 \cup \mathbf{C}^3/\mathbf{Z}_4 \cup \mathbf{C}^3/\mathbf{Z}_5}$$

Inhomogeneous coordinates:

$$(Y_1, Z_1, \varepsilon_1), (X_2, Z_2, \varepsilon_2), (X_3, Y_3, \varepsilon_3), (X_4, Y_4, Z_4).$$

In what follows, $(X_4, Y_4, Z_4) = (x, y, z)$.

$$\begin{cases} x = \varepsilon_1^{-3/5} = X_2 \varepsilon_2^{-3/5} = X_3 \varepsilon_3^{-3/5} \\ y = Y_1 \varepsilon_1^{-2/5} = \varepsilon_2^{-2/5} = Y_3 \varepsilon_3^{-2/5} \\ z = Z_1 \varepsilon_1^{-4/5} = Z_2 \varepsilon_2^{-4/5} = \varepsilon_3^{-4/5} \end{cases}$$

$$
\begin{array}{rcccccccc}
\mathbf{C}P^3(3,2,4,5) = & \mathbf{C}^3/\mathbf{Z}_3 & \cup & \mathbf{C}^3/\mathbf{Z}_2 & \cup & \mathbf{C}^3/\mathbf{Z}_4 & \cup & \mathbf{C}^3/\mathbf{Z}_5 \\
 & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
 & (Y_1, Z_1, \varepsilon_1), & & (X_2, Z_2, \varepsilon_2), & & (X_3, Y_3, \varepsilon_3), & & (x, y, z)
\end{array}
$$

Cellular decomposition

$$\mathbf{C}P^3(3,2,4,5) = \mathbf{C}^3/\mathbf{Z}_5 \cup \mathbf{C}P^2(3,2,4)$$

The first Painleve equation will be given on $\mathbf{C}^3/\mathbf{Z}_5$.
2-dim weighted proj. space $\mathbf{C}P^2(3,2,4)$ is attached at “infinity” .

A study of a singularity ($x=\infty$ or $y=\infty$ or $z=\infty$).

⇒ A study of the behavior around $\mathbf{C}P^2(3,2,4)$.

$$\mathbf{C}P^3(3,2,4,5) = \underset{\substack{\cup\!\!\!\!\cup \\ (Y_1, Z_1, \varepsilon_1),}}{\mathbf{C}^3/\mathbf{Z}_3} \cup \underset{\substack{\cup\!\!\!\!\cup \\ (X_2, Z_2, \varepsilon_2),}}{\mathbf{C}^3/\mathbf{Z}_2} \cup \underset{\substack{\cup\!\!\!\!\cup \\ (X_3, Y_3, \varepsilon_3),}}{\mathbf{C}^3/\mathbf{Z}_4} \cup \underset{\substack{\cup\!\!\!\!\cup \\ (x, y, z)}}{\mathbf{C}^3/\mathbf{Z}_5}$$

Give the (P1) on the fourth coord (x, y, z) .

In the other coordinates,

$$\begin{cases} \dfrac{dx}{dz} = 6y^2 + z \\[2ex] \dfrac{dy}{dz} = x, \end{cases} \qquad \begin{cases} \dfrac{dY_1}{d\varepsilon_1} = \dfrac{2Y_1(2Y_1^2 + Z_1/3) - 1}{5\varepsilon_1(2Y_1^2 + Z_1/3)} \\[2ex] \dfrac{dZ_1}{d\varepsilon_1} = \dfrac{4Z_1(2Y_1^2 + Z_1/3) - \varepsilon_1}{5\varepsilon_1(2Y_1^2 + Z_1/3)}, \end{cases}$$

$$\begin{cases} \dfrac{dX_2}{d\varepsilon_2} = \dfrac{3X_2^2 - 12 - 2Z_2}{5\varepsilon_2 X_2} \\[2ex] \dfrac{dZ_2}{d\varepsilon_2} = \dfrac{4X_2 Z_2 - 2\varepsilon_2}{5\varepsilon_2 X_2}, \end{cases} \qquad \begin{cases} \dfrac{dX_3}{d\varepsilon_3} = \dfrac{4 - 4Y_3^2 + 3X_3\varepsilon_3}{5\varepsilon_3^2} \\[2ex] \dfrac{dY_3}{d\varepsilon_3} = \dfrac{-4X_3 + 2Y_3\varepsilon_3}{5\varepsilon_3^2}, \end{cases}$$

(P1) is a rational ODE on $\mathbf{C}P^3(3,2,4,5)$.

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Thm. Any solutions of (P_I) are meromorphic.

For the proof, suppose that a sol. of (P_I) has a singularity at finite $z = z_*$;

$$x(z) \to \infty \text{ or } y(z) \to \infty \quad \text{as} \quad z \to z_*.$$

The coordinate change

$$\begin{pmatrix} X_2 \\ Z_2 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} xy^{-3/2} \\ zy^{-2} \\ y^{-5/2} \end{pmatrix}. \qquad \begin{matrix} X_2 \to 2 \\ Z_2 \to 0 \\ \varepsilon_2 \to 0. \end{matrix} \quad \text{as} \quad z \to z_*.$$

It is convenient to rewrite as a 3-dim dynamical system as

$$\begin{cases} \dfrac{dX_2}{d\varepsilon_2} = \dfrac{3X_2^2 - 12 - 2Z_2}{5\varepsilon_2 X_2} \\ \dfrac{dZ_2}{d\varepsilon_2} = \dfrac{4X_2 Z_2 - 2\varepsilon_2}{5\varepsilon_2 X_2}, \end{cases} \Longrightarrow \begin{cases} \dot{X}_2 = \frac{3}{2}X_2^2 - 6 - Z_2 \\ \dot{Z}_2 = 2Z_2 X_2 - \varepsilon_2 \\ \dot{\varepsilon}_2 = \frac{5}{2}\varepsilon_2 X_2. \end{cases}$$

This system has a fixed point $(X_2, Z_2, \varepsilon_2) = (2, 0, 0)$.

The solution converges to the fixed point:

$$(X_2, Z_2, \varepsilon_2) = (2, 0, 0). \quad J = \begin{pmatrix} 6 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{pmatrix}$$

Poincare's linearization theorem.

$$\frac{d^2y}{dz^2} = 6y^2 + z. \quad \Rightarrow \quad \begin{cases} \dot{X}_2 = \frac{3}{2}X_2^2 - 6 - Z_2 \\ \dot{Z}_2 = 2Z_2X_2 - \varepsilon_2 \\ \dot{\varepsilon}_2 = \frac{5}{2}\varepsilon_2 X_2. \end{cases}$$

$$\frac{d^2y}{dz^2} = 6y^2.$$

$$X = X_2 - 2.$$

$$\begin{cases} \dot{u} = 6u - v \\ \dot{v} = 4v + w \\ \dot{w} = 5w. \end{cases} \quad \xleftarrow{\text{Linearization}} \quad \begin{cases} \dot{X} = 6X - Z_2 + (\text{nonlinear}) \\ \dot{Z}_2 = 4Z_2 - \varepsilon_2 + (\text{nonlinear}) \\ \dot{\varepsilon}_2 = 5\varepsilon_2 + (\text{nonlinear}). \end{cases}$$

Normal form theory of dynamical systems

Linearization Theorem.(Poincare)

Holomorphic vec. field on $\mathbf{C}^n$

$$Jx + f(x), \quad f \sim O(||x||^2)$$

with the fixed point $x = 0$.

If eigenvalues of the Jacobi matrix J satisfy a certain algebraic condition, then $\exists$ local analytic coord. transformation near $x = 0$ s.t. $Jx + f(x)$ is transformed into the linear vec. field Jx.

Thm. $^{\exists}$local analytic transformation defined near each movable singularity s.t.
(P1) is transformed into the integrable Hamiltonian system
$y'' = 6y^2$

Cor. Any solutions of (P1) are meromorphic.

All Painleve equations are locally transformed into integrable equations near poles.
(necessary condition for the Painleve property)

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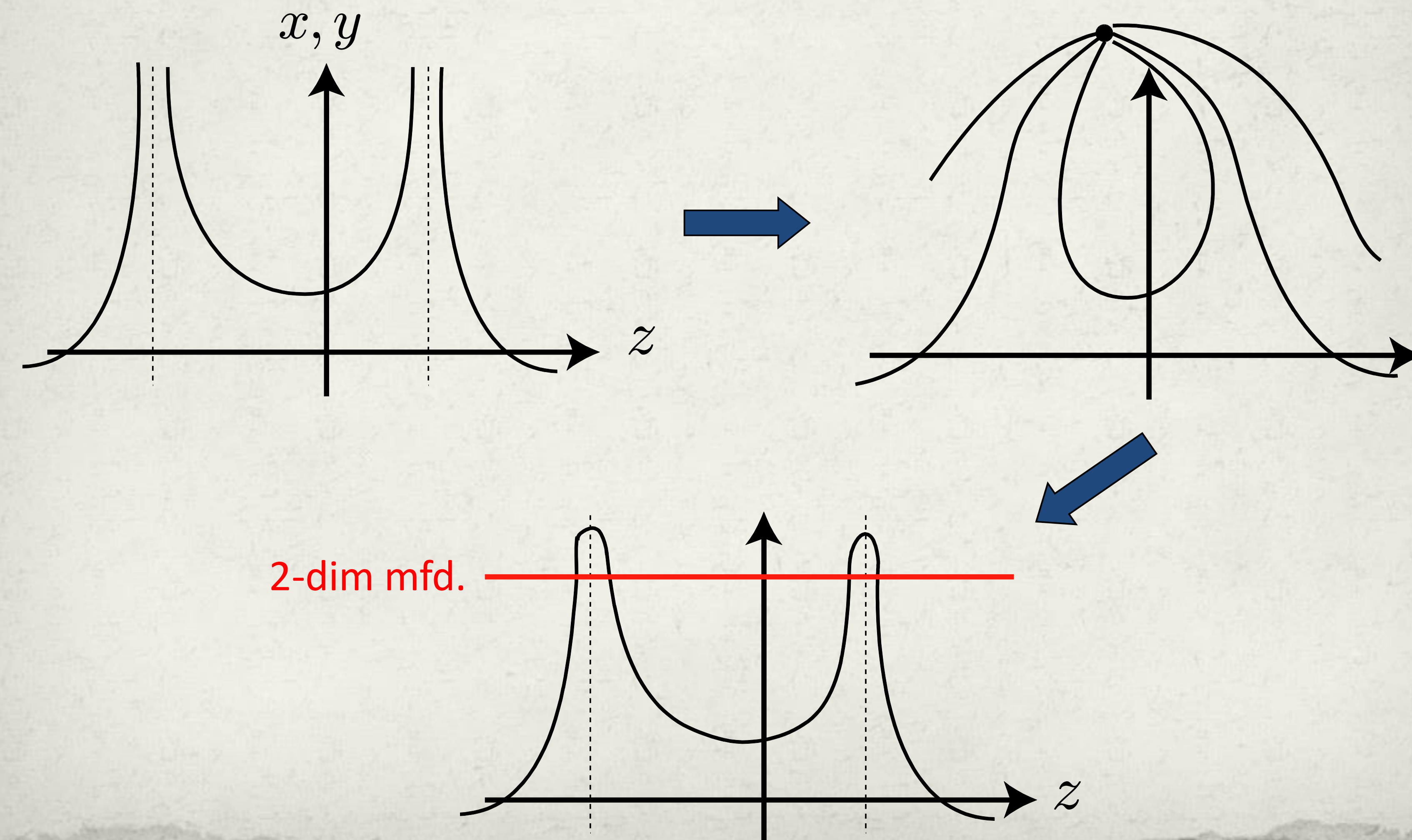
The fixed point $(X_2, Z_2, \varepsilon_2) = (2, 0, 0)$ is a singularity of the foliation defined by (P1).

$\Rightarrow$ resolution of sing. by a blow-up.



$$\begin{cases} \dfrac{dx}{dz} = 6y^2 + z \\ \dfrac{dy}{dz} = x, \end{cases} \quad \Longrightarrow \quad \begin{cases} \dot{X}_2 = \frac{3}{2}X_2^2 - 6 - Z_2 \\ \dot{Z}_2 = 2Z_2X_2 - \varepsilon_2 \\ \dot{\varepsilon}_2 = \frac{5}{2}\varepsilon_2 X_2. \end{cases}$$

$$\xrightarrow[\text{affine}]{} \quad \begin{cases} \dot{u} = 6u + (\text{nonlinear}) \\ \dot{v} = 4v + w + (\text{nonlinear}) \\ \dot{w} = 5w. \end{cases}$$

We introduce the weighted blow-up by

$$\begin{cases} u & = u_1^6 & = v_2^6 u_2 & = w_3^6 u_3 \\ v & = u_1^4 v_1 & = v_2^4 & = w_3^4 v_3 \\ w & = u_1^5 w_1 & = v_2^5 w_2 & = w_3^5 \end{cases}$$

The exceptional divisor is $\mathbf{C}P^2(6,4,5)$

$$\Longrightarrow \begin{cases} \dfrac{du_3}{dv_3} = \dfrac{1}{8}\left(v_3^2 w_3 + 3v_3 w_3^2 + 2w_3^3 - 8u_3 v_3 w_3^3 - 10u_3 w_3^4 + 12u_3^2 w_3^5\right) \\ \dfrac{dw_3}{dv_3} = \dfrac{1}{4}\left(4 + v_3 w_3^4 + w_3^5 - 2u_3 w_3^6\right). \end{cases}$$

$$(P_I) \Rightarrow \begin{cases} \dfrac{du}{dv} = \dfrac{1}{8}\left(v^2w + 3vw^2 + 2w^3 - 8uvw^3 - 10uw^4 + 12u^2w^5\right) \\ \dfrac{dw}{dv} = \dfrac{1}{4}\left(4 + vw^4 + w^5 - 2uw^6\right). \end{cases}$$

The coordinate transformation is given by

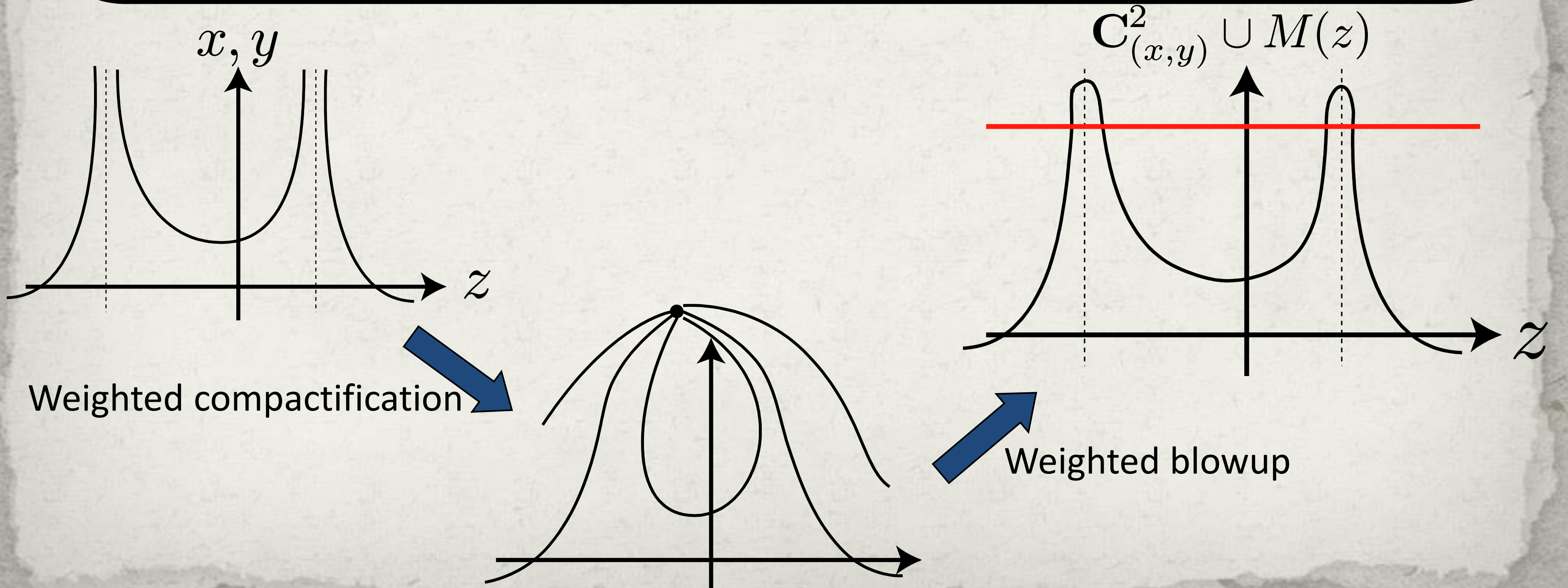
$$(*) \begin{cases} x = uw^3 - 2w^{-3} - \frac{1}{2}zw - \frac{1}{2}w^2 \\ y = w^{-2} \\ z = v. \end{cases}$$

- The independent value z is not transformed. $(*)$ defines a fiber bundle over z -space.
- $(*)$ is symplectic $-2du \wedge dw = dx \wedge dy$
- $\mathbf{C}^2_{(u,w)}/\mathbf{Z}_2$ is an algebraic surface $M(z)$ given by
  $$V^2 = UW^4 + 2zW^3 + 4W$$
- $(*)$ defines an symplectic alg. surface $\mathbf{C}^2_{(x,y)} \cup M(z)$.

Thm. The surface $\mathbf{C}^2_{(x,y)} \cup M(z)$ is a space of initial conditions for (P1).

i.e. any solutions of (P1) are holomorphic global sections of the fiber bundle $(\mathbf{C}^2_{(x,y)} \cup M(z)) \times \mathbf{C}_{(z)}$.

Conversely, if a given ODE is polynomial on $\mathbf{C}^2_{(x,y)} \cup M(z)$, then it is (P1).



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Consider the n-dim polynomial system on $\mathbf{C}^n$

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z),$$

and the truncated one;

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z).$$

(A1) The truncated system is quasi-homogeneous;
$\exists$ positive integers $(p_1, \cdots, p_n, r)$ s.t.

$$f_i(\lambda^{p_1} x_1, \cdots, \lambda^{p_n} x_n, \lambda^r z) = \lambda^{1+p_i} f_i(x_1, \cdots, x_n, z).$$

<u>Lemma.</u> The truncated system is invariant under the $\mathbf{Z}_s$ action $(s = r+1)$,

$$(x_1, \cdots, x_m, z) \mapsto (\omega^{p_1} x_1, \cdots, \omega^{p_m} x_m, \omega^r z), \quad \omega := e^{2\pi i/s}.$$

and has a Laurent series solution

$$x_i(z) \sim c_i (z - z_0)^{-p_i}.$$

Consider the polynomial system

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z),$$

and the truncated one;

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z).$$

(A1) $f_i(\lambda^{p_1} x_1, \cdots, \lambda^{p_n} x_n, \lambda^r z) = \lambda^{1+p_i} f_i(x_1, \cdots, x_n, z).$

(A2) $g_i(\lambda^{p_1} x_1, \cdots, \lambda^{p_n} x_n, \lambda^r z) = o(\lambda^{1+p_i}), \quad \lambda \to \infty,$

(A3) The full system is also invariant under the $\mathbf{Z}_s$ action.

$$\begin{aligned}(p_1, p_2, r) &= (2, 3, 4), \quad \text{(the first Painlevé)} \\ &= (1, 2, 2), \quad \text{(the second Painlevé)} \\ &= (1, 1, 1), \quad \text{(the fourth Painlevé).}\end{aligned}$$

For the system with (A1) to (A3),

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z),$$

assume the Laurent series solution

$$x_i(z) = c_i(z - z_0)^{-p_i} + a_{i,1}(z - z_0)^{-p_i+1} + a_{i,2}(z - z_0)^{-p_i+2} + \cdots$$

$\{c_i\}_{i=1}^n$ is a root of the equation $-p_i c_i = f_i(c_1, \cdots, c_n, 0)$.

Def. The Kovalevskaya matrix is defined by

$$K = \left\{ \frac{\partial f_i}{\partial x_j}(c_1, \cdots, c_m, 0) + p_i \delta_{ij} \right\}_{i,j=1}^n$$

Eigenvalues of K are called the Kovalevskaya exponents.
-1 is always the eigenvalue of K.

Laurent series solution

$$x_i(z) = c_i(z - z_0)^{-p_i} + a_{i,1}(z - z_0)^{-p_i+1} + a_{i,2}(z - z_0)^{-p_i+2} + \cdots$$

The coefficient $a_j = (a_{1,j}, \cdots, a_{m,j})^T$ satisfies

$$(K - jI)a_j = (\text{known number}).$$

<u>Case 1</u>. If j is not K-exp, a_j is uniquely determined.

<u>Case 2</u>. If j is one of the K-exp,

(2-a) no solution ➡ no Laurent series sol.

(2-b) $\exists$ solution ➡ a_j includes an arbitrary parameter.

Classical Painleve test.

If a given n-dim ODE has the Painleve property, then there exists a leading coeffi. $\{c_i\}_{i=1}^n$ s.t. all of the associated K-exp are positive integers (except for -1).

Consider the system with (A1) to (A3).

$$\frac{dx_i}{dz} = f_i(x_1, \cdots, x_n, z) + g_i(x_1, \cdots, x_n, z),$$

The system is well-defined on the weighted proj. sp.

$$M := \mathbf{C}P^{n+1}(p_1, \cdots, p_n, r, s) = \mathbf{C}^{n+1}/\mathbf{Z}_s \cup \mathbf{C}P^n(p_1, \cdots, p_n, r).$$

On each inhomogeneous coordinates, rewrite it as a n+1 dim autonomous vector field.

We will find fixed points on the "infinity" $\mathbf{C}P^n(p_1, \cdots, p_n, r)$.

Thm. The eigenvalues of the Jacobi matrix at the fixed point are given by

$\lambda = r, s,$ and n-1 Kovalevskaya exponents (except for -1).

Cor. The Kovalevskaya exponents are invariant under the action of Aut of the weighted proj. sp M.

Application: In Kawakami, Nakamura, Sakai (arXiv:1209.3836), there is a list of 4-dim Painleve equations. Among them,

$$H_{Gar}^{4+1} = p_1^2 - q_1^2 p_1 + p_2 q_1 q_2 - p_2 q_2^2 + p_1 p_2 + p_1 z - \beta q_1 + \alpha q_2,$$

$$H_{II}^{Mat} = \frac{1}{2} p_1^2 - q_1^2 p_1 - 4 p_2 q_1 q_2 - 2 p_2 q_2^2 + p_1 p_2 + p_1 z - \beta q_1 + \alpha q_2$$

We can conclude that they are actually different ODEs because K-exponents of them are different.

Both of them have 8 types of Laurent series. K-exp are

For H_{Gar}^{4+1},

$(4, 2, 1) \times 3$ (principle Laurent sol)

$(5, 4, -2) \times 5$ (non-principle)

For H_{II}^{Mat},

$(4, 2, 1) \times 3$

$(5, 4, -2) \times 2$

$(8, 4, -5) \times 2$

$(4, 4, -1) \times 1$

Thm. The system has n-para family of Laurent series sol, iff there exists a fixed point on the infinity set s.t.

(i) All e.values are positive integers (classical Painleve test).

(ii) The Jacobi matrix at the fixed point is semi-simple.

(iii) The system is locally linearizable via the normal form theory of dynamical systems.

If (i),(ii),(iii) hold, the singularity of the foliation at the fixed point is resolved by the weighted blow-up, whose weight is given by K-exp. On the blow-up space, the system is again a polynomial system.

Conjecture. 1 to 1 correspondence:

Painleve equations $\Longleftrightarrow$ (weight) + (K-exp)

Eq.	Weight (p,q,r,s)	K-exp	h
$P_1(E_8)$	$\mathbf{C}P^3(3,2,4,5)$	6	6
$P_2(E_7)$	$\mathbf{C}P^3(2,1,2,3)$	4	4
$P_4(E_6)$	$\mathbf{C}P^3(1,1,1,2)$	3	3
$P_3(D_8)$	$\mathbf{C}P^3(-1,2,4,1)$	2	2
$P_3(D_7)$	$\mathbf{C}P^3(-1,2,3,1)$	2	2
$P_3(D_6)$	$\mathbf{C}P^3(0,1,2,1)$	2	2
$P_5(D_5)$	$\mathbf{C}P^3(1,0,1,1)$	2	2
$P_6(D_4)$	$\mathbf{C}P^3(1,0,0,1)$	2	2

h : The weighted degree of the Hamiltonian.

Eq.	Weights $(p_1, q_1, \cdots, p_n, q_n, r, s)$	K-exp	h
$(P_1)_1$	$\mathbf{C}P^3(3,2,4,5)$	6	6
$(P_1)_2$	$\mathbf{C}P^5(5,2,3,4,6,7)$	8,5,2	8
$(P_1)_3$	$\mathbf{C}P^7(7,2,5,4,3,6,8,9)$	10,7,5,4,2	10

$(P_1)_n$ is the n-th member of the first Painleve hierarchy. (2n-dim Hamiltonian system).

Since $r = s - 1 = h - 2,$ minimal data for a weight is $(p_1, q_1, \cdots, p_n, q_n; h)$.

$$\begin{aligned}(p, q; h) &= (2,3;6) \quad \text{(first Painlevé)}\\ &= (1,2;4) \quad \text{(second Painlevé)}\\ &= (1,1;3) \quad \text{(fourth Painlevé)}\end{aligned}$$

Contents

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- **Newton diagram and weight**
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  - Painleve property

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- **Relation with singularity theory**

$(a_1, a_2, \cdots, a_{2n}; h)$ integers with $1 \cdot \; a_1 \cdot \; a_2 \cdot \; \cdots \cdot \; a_{2n} < h$.

Characteristic function

$$\chi(T) := \frac{(T^{h-a_1}-1)(T^{h-a_2}-1)\cdots(T^{h-a_{2n}}-1)}{(T^{a_1}-1)(T^{a_2}-1)\cdots(T^{a_{2n}}-1)}$$

Consider the following conditions:

(B1) $\chi(T)$ is polynomial.

(B2) $a_i + a_{2n-i-1} = h - 1$ for any i.

(B3) $a_1 =$ 1 or 2

<u>Lemma.</u> When $n = 1$, the weights satisfying the condition (B) are only $(2, 3; 6),\ (1, 2; 4),\ (1, 1; 3)$.
They are weights for (P_1),(P_2),(P_4), respectively.

Prop. When $n=2$, the weights satisfying the condition (B) are only

$$
\begin{aligned}
(a_1, a_2, a_3, a_4; h) = \; & (2,3,4,5;8), && H_{Gar}^{9/2} \\
& (2,2,3,3;6), && \text{Coupled P1} \\
& (1,2,3,4;6), && H_{Gar}^{7/2+1},\ H_{I}^{Mat} \\
& (1,2,2,3;5), && H_{Gar}^{5} \\
& (1,1,2,2;4), && H_{Gar}^{4+1},\ H_{II}^{Mat} \\
& (1,1,1,1;3), && H_{NY}^{A_4}
\end{aligned}
$$

For each weight, there exists the corresponding Painleve equation with a polynomial Hamiltonian. Some of them are listed in Sakai et. al (arXiv:1209.3836)).

Weight to Painleve (only 2-dim).

Step 1. Given (2,3;6), (1,2;4), (1,1;3), consider the generic quasi-homogeneous polynomials

$$H = c_1 x^3 + c_2 y^2$$

$$H = c_1 x^4 + c_2 x^2 y + c_3 y^2$$

$$H = c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3$$

Step 2. Simplify by the symplectic trans. and scaling.

$$H = x^3 + y^2$$

$$H = x^4 + y^2$$

$$H = x^2 y + x y^2$$

Weight to Painleve (only 2-dim).

Step 3. Versal deformation.

$$H = x^3 + y^2 + \underline{\alpha_4} x + \alpha_6$$

$$H = x^4 + y^2 + \underline{\alpha_2} x^2 + \alpha_3 x + \alpha_4$$

$$H = x^2 y + x y^2 + \underline{\alpha_1} x y + \alpha_2 x + \beta_2 y + \alpha_3$$

Step 4. Replace the parameter α by z if

$$\deg(\alpha) = \deg(H) - 2.$$

Step 5. Remove α if $\deg(\alpha) \neq \deg(H) - 1.$

$$H = x^3 + y^2 + zx$$

$$H = x^4 + y^2 + zx^2 + \alpha_3 x$$

$$H = x^2 y + x y^2 + zxy + \alpha_2 x + \beta_2 y$$

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