Cartan and Tanaka meet Pontryagin: from intrinsic geometry of distributions to extrinsic geometry of curves in flag varieties and back.

Lecture 1, March 7, 2024

§1 Very basic sketch on the Cartan's approach, Cartan connections and (moving) frames.

Lev Pontryagin physically met Eli Cartan in Moscow in 1935, where Cartan gave a lecture on his strategy for finding Betti numbers of classical compact Lie groups, using differential forms. Pontryagin set in the last row with a person who made the translation especially for him (note also that Pontryagin was blind from the age 14 after an accident) and probably Cartan and Pontryagin talked after the lecture. Shortly after this Pontryagin solved the problem discussed by Cartan using completely different ideas from the Morse theory.

In this notes we will discuss the surprising connection between Pontryagin's results in Optimal Control theory (the Pontryagin Maximum Principle) obtained much later (in the middle of 1950s) and the Cartan equivalence method (more precisely its algebraic version developed by Noboru Tanaka and its school). This relation was discovered by me in collaboration with Boris Doubrov, based on the previous work of Andrei Agrachev and Revaz Gamkrelidze.

Professor Morimoto already discussed intrinsic and extrinsic geometry. In some sense an extrinsic geometry is easier than the intrinsic one as in the former the action is of finite dimensional group versus the infinite dimensional group of diffeomorphisms in the latter. In my talks I am going to explain how for a wide class of geometric structures the intrinsic geometry can be studied by means of an extrinsic structures the intrinsic geometry can be studied by means of an extrinsic geometry of other objects intrinsically constructed from the original ones)

Very briefly speaking, the main ideas of Cartan were:

- in a class of geometric structures to choose the "most simple" homogeneous model, called the flat model
- to find the symmetry group G of the flat model

Denote by $G_+ \subset G$ the stabilizer of a point and let $\mathfrak{g} = \text{Lie}\, G$. G can be consider as a principle G_+-bundle over G/G_+ (the space of left cosets). G is endowed with the Maurer-Cartan form ω_G $\left(\omega_G(g)(X) = (L_{g^{-1}})_* X\right)$ satisfying

$$d\omega_G(X,Y) + [\omega_G(X), \omega_G(Y)] = 0$$ or, shortly,

$$d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$$

- any other structure can be considered as a deviation of the most simple one. To quantify this deviation one tries to assign to any structure from the considered class in a canonical way a principle G_+-bundle P of $\dim G$ and a $\mathfrak{g}$-valued 1-form ω (called the canonical Cartan connection of type (G, G^+)) satisfying natural property:

i) equivariancy $(R_a)^* \omega = \text{Ad}\, a^{-1} \omega \quad \forall a \in G^+$

ii) identifying each fiber of P with G^+ (by fixing

ii) identifying each fiber of P with G^+ (by fixing one point and using the action of G^+) the restriction of ω to the fiber coinsides with the Maurer Cartan form

iii) $\omega : T_pP \to \mathfrak{g}^+$ is an isomorphism $\forall p$

and in this assignment for the flat structure the bundle P is the canonical projection $G \to G/G^+$ and ω is the Maurer-Cartan form.

The deviation of the structure from the flat one is quantified by its curvature form Ω of the Cartan connection

$$\Omega := d\omega + \frac{1}{2}[\omega, \omega]$$

For example, for Riemannian metrics on an n-dimensional manifold M the flat structure is the flat Riemann metric $G = O(n) \ltimes \mathbb{R}^n$, $G^+ = O(n)$, P is the bundle of orthonormal frames on M

ω is (the Levi-Civita principal connection, the soldering form)

A Cartan connection ω defines a frame on bundle P (or a structure of absolute parallelism) compatible with the fibers of P, i.e. a collection of vector fields $\mathcal{F} = \{f_i\}_{i=1}^{\dim P} \subset \mathrm{Vec}\, M$ constituting basis of the tangent space T_pP for every $p \in P$ (this is the condition for a frame) such that a subcollection of $\mathcal{F}$ is tangent to the fibers of P and constitutes a frame on each fiber:

For this take a basis $\{e_i\}_{i=1}^{\dim P}$ in $\mathfrak{g}$ s.t. a subset of it is a basis in $\mathfrak{g}^+$ and take $\{\omega^{-1}(p)\{e_i\}\}_{i=1}^{\dim P}$

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Moving frame coming from Cartan connections satisfy special equivariant properties
In general it is not always possible to assign to geometric structure a canonical Cartan connection, but even if one just assigns a canonical moving frame it also solves the equivalence problem in principle.
In the classical differential geometry such as Riemannian, conformal, etc, there is no additional filtration of the tangent bundle. This led to the theory of prolongations of G-structures (Chern, Kobayashi, Sternberg) Noboru Tanaka ingeniously extended the theory of G-structures to the filtered structures, developing the nilpotent differential geometry.

Geometric structures of our interest are distributions
subbundles of the tangent bundle of a manifold M
without or with additional structures
on their fibers. Additional structures might be
inner products on each fiber (sub-Riemannian
metrics, so Riemannian metrics form a particular
case, when $D=TM$), complex structures on
each fiber (almost CR or CR structures) etc

Such structure are also called filtered
structures. In the classical differential such as the Riemannian and conformal geometry, there is no additional filtration of the tangent bundle. This led to the theory of prolongations of G-structures (Chern, Kobayashi, Sternberg). Noboru Tanaka ingeniously extended the theory of prolongations of G-structures to very wide class of filtered structures of constant type, developing the nilpotent differential geometry.

the nilpotent differential geometry.

Question: How to choose classes of distributions such that all distributions from the class are deviations from the (unique) flat one in this class?

§2 Tanaka nilpotent geometry

1) A distribution D induces the filtration by taking iterative Lie brackets

$$D^1 = D, \quad D^j = D^{j-1} + [D, D^{j-1}] \text{ (or, equivalently, } D^j = [D, D^{j-1}])$$

D is bracket-generating, if $\forall q \; \exists \mu$ s.t. $D^\mu(q) = T_q M$

2) Filtration: $D(q) \subset D^2(q) \subset \dots \subset D^\mu(q) = T_q M$

$(\dim D(q), \dim D^2(q), \dim D^3(q), \dots)$ is called the small growth vector, shortly s.g.v.

Equiregularity: $\exists$ a neighb U of q s.t. $\dim D^j(q)$ are constant for every j.

$$m(q) := \underbrace{D(q)}_{\mathfrak{g}_{-1}(q)} \oplus \underbrace{D^2(q)/D(q)}_{\mathfrak{g}_{-2}(q)} \oplus \dots \oplus \underbrace{D^\mu(q)/D^{\mu-1}(q)}_{\mathfrak{g}_{-\mu}(q)}$$

has the structure of graded Lie algebra ($\mathbb{Z}$-graded $\Rightarrow$ nilpotent)

$$\mathfrak{g}_i(q) = D^{-i}(q)/D^{-i+1}(q), \quad i<0$$

Lie brackets $\quad [X_i, X_j] := [\tilde{X}_i, \tilde{X}_j](q) \mod D^{-i-j+1}(q)$

$(\mathfrak{g}_i, \mathfrak{g}_j)$

Lie brackets of vector fields

the extension of X_i to the section of D^i

$m(q)$ is called the Tanaka symbol of the distribution D

From bracket genericity it is a $\mathbb{Z}_-$-graded Lie algebra generated by $\mathfrak{g}_{-1}$ (such algebra is called fundamental) by $\mathfrak{g}_{-1}$ (such algebra is called fundamental)

Examples I

a) $D = TM$, then $\mathfrak{m} = \mathbb{R}^n$, the commutative Lie algebra

b) D is a contact distribution, i.e. a corank 1 distribution on a manifold of dimension $2n+1$, s.t. if $D = \ker\alpha$ for a 1-form α (the defining form), then $d\alpha|_D$ is nondegenerate ($\Leftrightarrow \alpha \wedge (d\alpha)^n \neq 0$). In this case $\mathfrak{m} \cong (2n+1)$-dimensional Heisenberg algebra

A distribution D is said to have a constant Tanaka symbol $\mathfrak{m}$ (is of constant type $\mathfrak{m}$) if $\forall q$ $\mathfrak{m}(q) \cong \mathfrak{m}$ $\rightsquigarrow$ the class of objects for which one can apply the scheme above

We can define $\mathrm{Aut}(\mathfrak{m})$-principle bundle $P^0 \to M$ (here $\mathrm{Aut}(\mathfrak{m})$ is the group of automorphisms of $\mathfrak{m}$ preserving the grading)

$$P^0(\mathfrak{m}) = \{(q,\varphi) : \varphi : \mathfrak{m} \to \mathfrak{m}(q) \text{ is a graded Lie algebra isomorphism}\}$$

Additional structures on a distribution D of constant type $\mathfrak{m}$ can be encoded as a reduction of the bundle $P^0(\mathfrak{m})$ with the structure group $G_0 \subset \mathrm{Aut}(\mathfrak{m})$. If $\mathfrak{g}_0 = \mathrm{Lie}\, G_0 = \mathrm{Der}\,\mathfrak{m}$

Such structures are called of constant type $(\mathfrak{m}, \mathfrak{g}_0)$, $\mathfrak{g}_0 \subset \mathrm{Der}\,\mathfrak{m}$ (in fact $\mathfrak{m} \oplus \mathfrak{g}^0$ has a natural graded Lie algebra structure of the semi-direct sum $\mathfrak{m} \rtimes \mathfrak{g}_0$)

3) • The flat distribution $D_{\mathfrak{m}}$ of type $\mathfrak{m}$ → the construction via Lie group theory:

Let $M(\mathfrak{m})$ be the simply connected Lie group s.t. $\mathrm{Lie}\, M(\mathfrak{m}) = \mathfrak{m}$

$D_{\mathfrak{m}}$ be the left-invariant distribution s.t.

$D_{\mathfrak{m}}(e) = \dots$

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$$D_{\mathfrak{m}}(e) = \mathfrak{g}_{-1}$$

• The flat structure of type $(\mathfrak{m}, \mathfrak{g}_0)$ is a left-invariant reduction of G^0-reduction $P \to M(\mathfrak{m})$ of $P^0 \to M(\mathfrak{m})$, i.e. s.t. the fibers of P are preserved by left-translations of $M(\mathfrak{m})$

4) The first significant Tanaka observation:

Theorem 1 (N. Tanaka, 1970)

The algebra of infinitesimal symmetries $\mathrm{symm}(D_{\mathfrak{m}})$, if finite dimensional, is isomorphic to the maximal nondegenerate $\mathbb{Z}$-graded Lie algebra $\mathfrak{u}(\mathfrak{m}) = \bigoplus_{i \geq 0} \mathfrak{g}_i$ so that its $\mathbb{Z}_-$-graded part is $\mathfrak{m}$, called the universal algebraic prolongation of $\mathfrak{m}$.

Nondegenerate means that nonzero $x \in \mathfrak{u}_{\geq 0}(\mathfrak{m})$, $\mathrm{ad}\, x|_{\mathfrak{m}} \neq 0$. (this nondegeneracy condition also called transitivity)

Rem: In the case when $\dim \mathfrak{u}(\mathfrak{m}) = \infty$ the algebra of formal Taylor series of infinitesimal symmetries

Note that $\quad \mathfrak{g}_0 = \mathrm{Lie\,Aut}(\mathfrak{m}) = \mathrm{Der}(\mathfrak{m})$

Similarly, the flat structure of type $(\mathfrak{m}, \mathfrak{g}_0)$ has the algebra of infinitesimal symmetries isomorphic to the maximal nondegenerate $\mathbb{Z}$-graded Lie algebra $\mathfrak{u}(\mathfrak{m}, \mathfrak{g}_0)$, so that its $\mathbb{Z}_+$-graded part is $\mathfrak{m} \oplus \mathfrak{g}_0$ (more precisely $\mathfrak{m} \rtimes \mathfrak{g}_0$)

Rem In the case when $\dim U(\mathfrak{m})$ is infinite dimensional the algebra of formal Taylor series of $\mathrm{symm}(D_{\mathfrak{m}}$ is isomorphic to the direct product $\overline{\mathfrak{u}(\mathfrak{m})} := \prod \mathfrak{g}_i$

the algebra of formal Taylor series of symm $\mathbb{D}_m$ is isomorphic to the direct product $\overline{u(m)} := \prod_{i \in \mathbb{Z}} g_i$

(see Chapter 6 of Tanaka 1970, ref. [7] of the abstract)

5) Theorem 2 (N. Tanaka, 1970)

c) If $\dim u(m) < \infty$ then to any distribution of constant type m on manifold M one can assign a canonical frame on a bundle of $\dim = \dim u(m)$. The flat distribution D_m of constant type m is the unique, up to local equivalence, among all distributions of type m with the algebra of infinitesimal symmetries has dimension $= \dim u(m)$ (and this is the maximal possible dimension)

(The existence of canonical Cartan connection is not guaranteed in general, only under some additional & rather restrictive assumptions)

In slightly more details (explained in much more details in Prof. Morimoto talk) if $u(m) = \bigoplus_{i \in \mathbb{Z}} g_i$

One constructs a chain of bundle

$$M \leftarrow P^0 \leftarrow P^1 \leftarrow P^2 \leftarrow \ldots \leftarrow P^k$$

where P^0 is $Aut(m)$ – principle bundle

$P^0 = \{ (q, \varphi) : \varphi : m \to m(q)$ is a graded Lie algebra isomorphism $\}$

and for $i > 0$ $P^i \to P^{i-1}$ is the affine bundle with the fibers being affine spaces over g_i

with the fibers being affine spaces over $\mathfrak{g}_i$ and $k = \#$ of nontrivial $\mathfrak{g}_i - 1$ so that P^k is endowed with the canonical frame

§3 The space of Tanaka symbols might be large.

In this approach one must to fix the symbol $\mathfrak{m}$ and consider distributions of constant type $\mathfrak{m}$.

In general, the space of all fundamental graded nilpotent Lie algebras, up to isomorphism, with fixed $\dim \mathfrak{m}$ and $\dim \mathfrak{g}_{-1}$ might be huge (impossible to classify) and also may depend of continuous parameters, see examples below.

<u>Example 1</u> The case of rank 2 distributions with small growth vector $(2,3,5,\ldots)$ (The reason for this assumption we always can reduce it or have a trivial local geometry near generic point)

a) $\dim M = 5$ $\to$ only one Tanaka symbol

$\mathfrak{m}$: 3 step truncated free Lie algebra with 2 generators

$\mathfrak{g}_{-1} = \langle X_1, X_2 \rangle$, $\mathfrak{g}_{-2} = \langle X_3 \rangle$, $\mathfrak{g}_{-3} = \langle X_4, X_5 \rangle$

The only nontrivial brackets are

$$[X_1, X_2] = X_3,\ [X_1, X_3] = X_4,\ [X_2, X_3] = X_5 \qquad (1)$$

The universal algebraic prolongation:

$\mathfrak{u}(\mathfrak{m}) \cong$ (split real form of) G_2 — 14-dimensional

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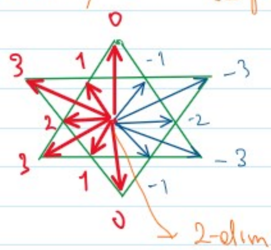


$\circ \Rrightarrow \otimes$

2-dim Cartan subalgebra $\subset \mathfrak{g}^0$

b) $\dim M = 6$ — 3 non isomorphic symbols

$D \cong D^3/D^2$ canonically up to a scaling

by the natural map (induced by Lie brackets of vector fields)

$$D \wedge \underbrace{D^2/D}_{1\text{-dim}} \to D^3/D^2 \qquad (2)$$

The identification (2) defines the canonical bilinear form, up to a scaling

$$B : D \times D \to \underbrace{D^4/D^3}_{1\text{-dim}}$$

via the natural map

$$D \wedge \underset{\substack{\wr \text{ via } (2) \\ D}}{D^3/D^2} \to D^4/D^3$$

and B is symmetric by Jacobi identities

$$\Big(\text{as } \big[X,[Y,[X,Y]]\big] = \big[Y,[X,[X,Y]]\big] \Big)$$

3 inequivalent symbols depending on signature of B, i.e. one can choose a basis X_1, X_2 of $\mathfrak{g}_{-1}$ s.t. B is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \text{ where } \varepsilon \in \{\underbrace{-1}_{\substack{\text{hyperbolic} \\ \text{symbol}}}, \underbrace{0}_{\substack{\text{parabolic} \\ \text{symbol}}}, \underbrace{1}_{\substack{\text{elliptic} \\ \text{symbol}}}\}$$

or,

$\mathfrak{g}_{-1} = \langle X_1, X_2 \rangle, \mathfrak{g}_{-2} = \langle X_3 \rangle, \mathfrak{g}_{-3} = \langle X_4, X_5 \rangle, \mathfrak{g}_{-4} = \langle X_6 \rangle$

s.t. in addition to relations (1)

$[X_1, X_4] = X_6, \quad [X_2, X_5] = \varepsilon X_6$

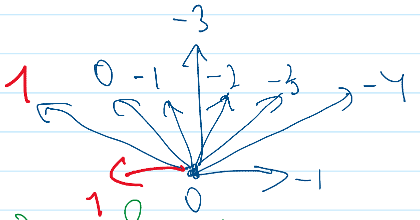
The universal algebraic prolongations

Elliptic & Hyperbolic $\to \dim \mathfrak{u}(\mathfrak{m}) = 8, \ \mathfrak{g}^1 = 0$

Parabolic $\to \dim \mathfrak{u}(\mathfrak{m}) = 11,$

$\dim \mathfrak{g}^0 = 3, \ \dim \mathfrak{g}^1 = 2, \ \mathfrak{g}^2 = 0$

Visualization:



c) $\dim M = 7 \to 8$ nonisomorphic symbols

c.1) The s.g.v is $(2,3,5,7)$

Then $B : \mathrm{Sym}^2 D \to \underbrace{D^4/D^3}_{\text{2-dimensional}}$

$\Downarrow$

$B^* : (D^4/D^3)^* \to \mathrm{Sym}^2 D^* \to$ the space of symmetric forms on D

defines a plane is $\mathrm{Sym}^2 D^*$

$\to$ again 3 symbols depending on how this plane intersects the cone of rank 1 forms in $\mathrm{Sym}^2 D^*$

c.2) The s.g.v is $(2,5,6,7,8)$

$\mathcal{B} : \mathrm{Sym}^2 D \to \underbrace{D^4/D^3}_{\text{1-dim}} \to$ a symmetric form on D, upto a scaling

$\to$ 3 case based on the signature

$d : \underbrace{D \wedge D^4/D^3}_{\text{1-dim}} \to \underbrace{D^5/D^4}_{\text{1-dim}} \to$ a linear form on D, upto a scaling

Moduli space of symbols $\cong$ moduli space of pairs $(\mathcal{B}, d)$ with $B = 0$ & $d = 0 \to$ 6 equivalence classes

(B is sign-definite $\to$ 1 equivalence class

B is sign-indefinite $\to$ 2 equivalence classes according to whether or not $\ker d$ is B-isotropic

B is degenerate $\to$ 2 equivalence classes according to whether or not $\ker d$ is B-isotropic

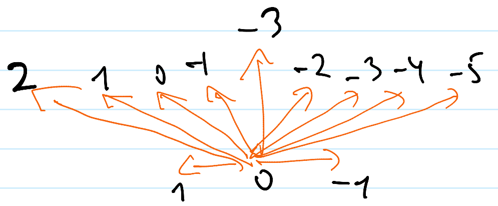
The most symmetric case for $\dim M = 7$

The most symmetric case for $\dim M = 7$ is from cases in c2) with B degenerate and ker d being B-isotropic.

$\dim \mathfrak{u}(m) = 13 \qquad \dim g_0 = 3, \dim g_1 = 2, \dim g_2 = 1$

$g_3 = 0$

Visualization



d) $\dim M = 8 \to$ continuous parameters (moduli) appear

Assume that s.g.v. is $(2,3,5,6,7,8)$

Similar to before we have

$B : \mathrm{Sym}^2 D \to \underbrace{D^4/D^3}_{1\text{-dim}} \to$ a symmetric bilinear form, up to scaling

$d_1 : D \wedge \underbrace{D^4/D^3}_{1\text{-dim.}} \to \underbrace{D^5/D^4}_{1\text{-dim}} \to$ a linear form, up to scaling

$d_2 : D \wedge \underbrace{D^5/D^4}_{1\text{-dim}} \to \underbrace{D^6/D}_{1\text{-dim}} \to$ another bilinear form, up to scaling

If B is sign-indefinite and the following 4 lines in D are distinct, i.e. 4 distinct lines on the projective line $\mathbb{P}D$:

2 isotropic lines of B, ker d_1 & ker d_2.

Then we can take the cross-ratio of them

$$[d_1, d_2, d_3, d_4] = \frac{d_1 - d_2}{d_3 - d_2} : \frac{d_1 - d_4}{d_3 - d_4}$$

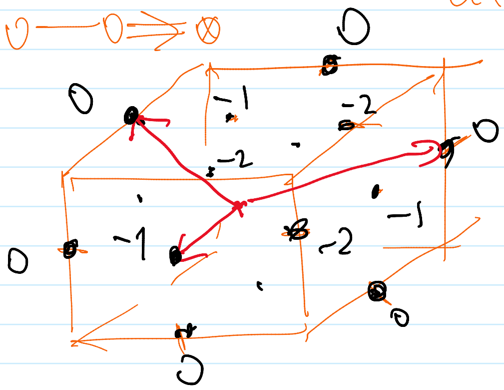
Example 2 The case of rank 3 distributions with s.g.v $(3, 6, \ldots)$

s.g.v (3,6,...)

a) $\dim M = 6$ → only one Tanaka symbol:
step 2 truncated free Lie algebra with 3 generators

$\mathfrak{g}_{-1} = \langle X_1, X_2, X_3 \rangle$, $\mathfrak{g}_{-2} = \langle [X_1,X_2], [X_1,X_3], [X_2,X_3] \rangle$

$\mathfrak{u}(\mathfrak{m}) = \mathfrak{so}(4,3)$



$\mathfrak{g}_0 \sim \mathfrak{gl}_3(\mathbb{R})$

b) $\dim M = 7$

The Lie brackets define the isomorphism (the Levi map) $\mathcal{L} : \Lambda^2 D \to D^2/D$

by $\mathcal{L}(X \wedge Y) = [X,Y]$

On the other hand $\mathbb{P}\Lambda^2 D \sim \mathbb{P} D^*$

$X \wedge Y \to [\alpha] \in \mathbb{P} D^*$ s.t $\ker \alpha = \langle X \wedge Y \rangle$

Choosing a volume form Ω on D this defines a linear map
$\perp : \Lambda^2 D \to D^*$; $X \wedge Y \to \alpha$ s.t $\ker \alpha = \langle X \wedge Y \rangle$ and
$\alpha(Z) = 1$ if $\Omega(X \wedge Y \wedge Z) = 1$

This map is defined up to scaling (once one changes the volume form)

Composing $\perp$ and $\mathcal{L}$ we get that $D^2/D \sim D^*$ $(*)$

On the other hand, there is a natural map
$A : D \wedge D^2/D \to \underbrace{D^3/D^2}_{1\text{-dim}}$ which is not identically zero

and it defines a map from D^2/D to D^*,
up to a scaling. Since by $(*)$ $D^* \sim D^2/D$

and it defines a map from $D/_{D}$ to D^*,
up to a scaling. Since by (*) $D^* \sim D^2/_{D}$
we have a well defined nonzero linear map, up to
a scaling, from $D^2/_{D}$ to itself $\rightarrow$ so the space
of Tanaka symbols $\cong \mathbb{P}\,\mathrm{End}(D^*) \rightarrow$ contains moduli.

Question: Can we make a unified construction of canonical moving frame independently of the Tanaka symbol and based on another more rough basic invariant of a distribution which is easily classifiable and has a discrete classification?

We will answer this question in the next lecture