

Cartan and Tanaka meet Pontryagin:
from intrinsic geometry of distributions
to extrinsic geometry of curves in flag
varieties and back.

Lecture 2, March 8 2024.

In the first lecture we mainly focus on Cartan's and Tanaka's contributions. Today we will start with Pontryagin and then on how the ideas of these three giants meet.

§4 Symplectification procedure

§4.1 Abnormal extremals / singular curves.

A completely different idea, taking its origin in Optimal Control, based on the fact that most of distributions have plenty of special curves, called singular curves (or abnormal extremal trajectories in Optimal Control) playing the role similar to geodesics in Riemannian geometry and one can study the geometry of distributions via studying the path geometry of these curves.

First we will describe this completely different idea and then using this idea will return to the Tanaka like constructions.

Basics on optimal control (very briefly)

Consider the following Optimal Control problem with the distribution $\mathcal{D}$ being a constraint:



$$J(q(t)) = \int_0^t L(q, \dot{q}) dt \rightarrow \min$$

$\dot{q}(t) \in \mathcal{D}(q(t))$ a.e. $\rightarrow$ for example $\|\dot{q}\|^2$ w.r.t. some Euclidean norm of fibers of $\mathcal{D}$

$q(0) = q_0, q(T) = q_1$

(3)

q_0

$q(0) = q_0, q(T) = q_1$ fibers of D

$$q(0) = q_0, q(T) = q_1$$

Pontryagin Maximum principle

$$h(\underbrace{p}_{T_q^*M}, \underbrace{q}_M, \underbrace{v}_{D(q)}) = p(v) + v L(q, v)$$

If a curve $q(t)$ is a minimizer of the problem then there exist $(v, p(t)) \neq 0$ ($v \leq 0$) is constant & $p(t) \in T_{q(t)}^*M$ and a "control" $v(t) \in D(q(t))$ s.t.

1) Maximality condition:

$$h(p(t), q(t), v(t), v) = \max_{v \in D(q(t))} p(v) + v L(q, v) \text{ a.e. } t \quad (4)$$

2) Hamiltonian system: if $\lambda(t) = (p(t), q(t))$ then it satisfies the following Hamiltonian system

$$\dot{\lambda}(t) = \vec{h}(\lambda(t), v(t), v) \text{ a.e. } t \quad (5) \Leftrightarrow$$

$$\begin{cases} \dot{q} = \frac{\partial h}{\partial p} \\ \dot{p} = - \frac{\partial h}{\partial q} \end{cases}$$

The curves $\lambda(\cdot) (q(\cdot))$ in the cotangent bundle T^*M (or M), satisfying (4) and (5) are called the Pontryagin extremals (the Pontryagin extremal trajectories) of (3).

Note that the case $v=0$ is possible and in this case the Pontryagin extremals are called abnormal,

if $v=0$ the cost L does not appear in (4) and (5), so the abnormal extremals depend on the distribution D only and not on the cost.

From (5) for $v=0$ it follows that $p(t)$ is such that

$$\max_{v \in D(q(t))} p(t)(v) \text{ is attained} \Rightarrow$$

$$p(t) \big|_{D(q(t))} = 0 \text{ so it means}$$

that the abnormal extremal must lie on the annihilator $D^\perp$ of D , where

$$D^\perp = \{(p, q) \in T^*M : p(D(q)) = 0\}$$

The geometric definition of the abnormal extremals on the projectivization $\mathbb{P}T^*M$ of T^*M .

Recall that T^*M is endowed with the canonical symplectic structure $\epsilon = ds$, where s is the tautological (the Liouville) 1-form on T^*M :

$$\begin{array}{ccc} T^*M & \text{For } \lambda = (p, q) \in T^*M & \\ \pi \downarrow & s(\lambda)(v) := p(d\pi(v)) \quad \forall v \in T_\lambda T^*M & \\ M & \text{In canonical coordinates } (q^1, \dots, q^n, p_1, \dots, p_n) \text{ on } T^*M & \\ & s = p_i dq^i \text{ and } ds = dp_i dq^i & \end{array}$$

The curve $\lambda(\cdot)$ in T^*M is an abnormal extremal if

- 1) $\lambda(\cdot) \subset D^\perp$ (maximality condition)
- 2) $\lambda'(t) \in \ker(\epsilon|_{D^\perp})$ (the condition on Hamiltonian system)

$$\text{Let } W_D = \{ \lambda \in (D)^\perp : \ker(\epsilon|_{D^\perp}) \neq 0 \}$$

the degeneracy locus of D

If $\lambda(\cdot)$ is an abnormal extremal which is also immersed ($\lambda'(t) \neq 0 \forall t$) then from 1) & 2) we must have

$$\lambda(t) \in W_D \quad \forall t$$

Examples

- If $\text{rank } D$ is odd then $W_D = D^\perp$

Indeed, $\dim D^\perp$ is odd $\Rightarrow \ker \epsilon|_{D^\perp} \neq 0$

- If $\text{rank } D = 2$ then $W_D = (D^\perp)^\perp$
the annihilator of D

- More generally if $\text{rank } D = 2k$, then $\forall q \in M$

$W_D \cap D^\perp(q)$ is a codim 1 algebraic variety of degree k in $D^\perp(q)$ given by the Pfaffian of certain $2k \times 2k$ skew-symmetric matrix (the Goh matrix) with entries linear w.r.t. the fibers (of $D^\perp$)

In all cases $\dim W_D$ is odd

Let $\hat{C}(\lambda) := \ker \epsilon|_{W_D}$ - the characteristic distribution on W_D

$\text{rank } \hat{C}(\lambda)$ is odd but in general varies from point to point

$$W_D^{\text{reg}} = \left\{ \lambda \in W_D : \dim \hat{C}(\lambda) \text{ is constant in a neigh of } \lambda \right\}$$

$$W_D^{\text{reg}} = \left\{ \lambda \in W_D : \hat{C}(\lambda) \text{ is involutive (completely integrable)} \right\}$$

Rem In a neighb. of $\lambda \in W_D^{\text{reg}}$ $\hat{C}$ is involutive (completely integrable) as a Cauchy characteristic of a (degenerate) closed 2-form

$$\text{Let } W_D^1 = \{ \lambda \in W_D : \dim \hat{C}(\lambda) = 1 \} \subset W_D^{\text{reg}}$$

For simplicity, we restrict ourselves to the case when

$$W_D^{\text{reg}} = W_D^1$$

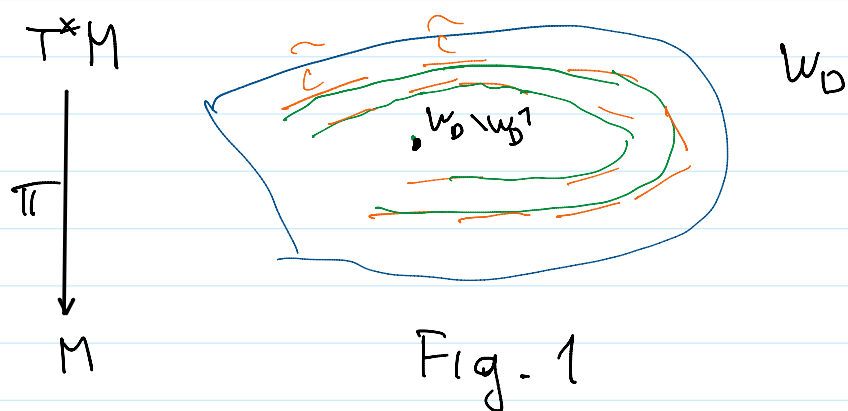
For example,

$$\cdot \text{ if } \text{rank } D = 3 \quad W_D^1 = D^\perp \setminus (D^2)^\perp$$

$$\cdot \text{ if } \text{rank } D = 2 \quad W_D^1 = (D^2)^\perp \setminus (D^3)^\perp$$

so in these cases if D is bracket generating then

$$W_D^{\text{reg}} = W_D^1 \text{ and it is an open and dense set of } W_D$$



The unparametrized abnormal extremal through $\lambda \in W_D^1$ is an integral curve of the line distribution D , called a regular abnormal extremal.

Rem If $W_D^{\text{reg}} \neq W_D^1$ the theory below will still work, just instead of curves in flag varieties we will get submanifolds of dimension = $\dim \hat{C}$.

For the sequel, it is more convenient to work with the projectivization of $\mathbb{P}T^*M$ rather than T^*M

The projectivization $\mathbb{P}T^*M$ of T^*M is the bundle over M with the fiber over q equal to the projective space $\mathbb{P}T_q^*M$ of the cotangent space at q .

$$\begin{array}{c} \mathbb{P}T^*M \\ \downarrow \pi \\ T^*M \end{array}$$

The tautological 1-form s on T^*M induces the contact distribution $\hat{\Delta} = \pi_* \ker s$ on $\mathbb{P}T^*M$ which in turn induces

$\Delta = \pi_* \omega_{\mathcal{L}}$
 on $\mathbb{P}T^*M$, which in turn induces
 the even-contact (the quasi-contact) distribution $\hat{\Delta}$
 on $\mathbb{P}W_D^1$.

Then $C := \pi_* \hat{C}$ on $\mathbb{P}W_D^1$ is the Cauchy
 characteristic of $\hat{\Delta}$, i.e. it is the maximal
 subdistribution of $\hat{\Delta}$ such that
 $[C, \hat{\Delta}] \subset \hat{\Delta}$.

So, we have the figure similar to Fig. 1
 with $T^*M, W_D, W_D^1, \hat{C}$ replaced by
 $\mathbb{P}T^*M, \mathbb{P}W_D, \mathbb{P}W_D^1, C$, respectively.

The integral curves of the line distribution
 C will be called regular abnormal extremals
 as well.

§ 4.2. Jacobi curves of regular abnormal extremals

$\mathbb{P}W_D^1$ has two structures

- 1) The foliation by regular extremals
- 2) The fiber bundle over M

One can study the "dynamics" of the fibers of 2)
 along a leaf γ of 1). Roughly speaking, the Jacobi curve of γ
 describes this dynamics

In more details, let

- γ be a segment of abnormal extremal,
- O_γ be a neighborhood of γ in $\mathbb{P}W_D^1$ so that

$$\mathcal{N} = O_\gamma / \{ \text{the foliation of regular } \gamma \text{ abnormal extremals} \}$$

is a well-defined smooth manifold.

- $\varphi: O_\gamma \rightarrow \mathcal{N}$ be the canonical projection
 to the quotient manifold.

Then, as C is the Cauchy characteristic of $\tilde{\Delta}$
its flow preserves $\tilde{\Delta} \Rightarrow$

$\Delta := \varphi_* \tilde{\Delta}$ is a well-defined distribution on N and

it is a contact distribution $\Rightarrow$ a symplectic form is
defined, up to a scaling,
on each fiber of Δ

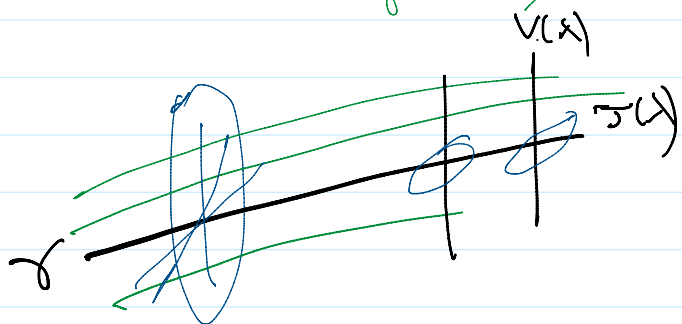
Now if $\pi: W_D \rightarrow M$, let

• $J = \pi^* D \Leftrightarrow (J) := \{v \in T_\lambda W_D : \pi_* v \in D(\pi(\lambda))\}$
 $\hookrightarrow$ the lift of D to W_D

• $V = \ker \pi_* \rightarrow$ the vertical distribution on W_D , i.e.
the distribution tangent w.r.t. to the fibers

$\forall \lambda \in \gamma$

$$J_\gamma(\lambda) := d\varphi(J(\lambda)) \subset \Delta(\gamma)$$



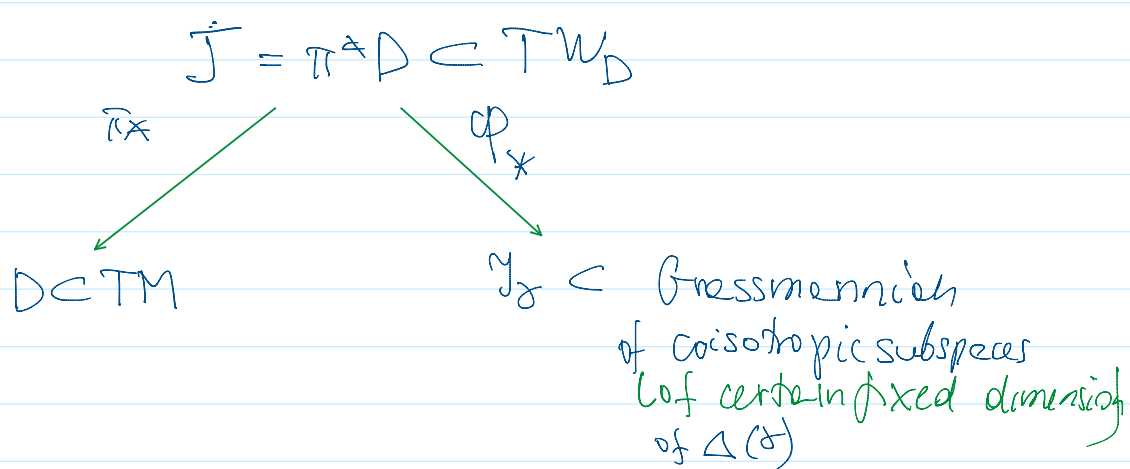
$\downarrow$ a slice representing $W = \mathcal{O}$ / the foliation of
regular abnormal
extremals

J_γ is an (unparametrized) curve of (anisotropic w.r.t
to the symplectic
structure on $\Delta(\gamma)$)

subspaces of $\Delta(\gamma) \subset T_\gamma N$, called the (reduced) Jacobi
curve of the abnormal extremal γ .

Coisotropic means the skew-symmetric complement $J_\gamma^\perp$ of J_γ lies
in J_γ : $(J_\gamma)^\perp \subset J_\gamma$

The double fibration diagram



of certain fixed dimension
of $\Delta(\mathfrak{g})$



One can build the osculating flag to
get the curve of symplectic (i.e. isotropic/coisotropic)
flags, called the (extended) Jacobi curve:

$$O_{sc}(\gamma_x) := \dots \subset \underbrace{\left(\gamma_x^{(0)}\right) \subset \left(\gamma_x^{(1)}\right) \subset \left(\gamma_x^{(2)}\right)}_{\text{isotropic}} \subset \gamma_x \subset \underbrace{\gamma_x^{(1)} \subset \gamma_x^{(2)} \subset \dots}_{\text{coisotropic}} \subset \Delta(\mathfrak{g}) \quad (*)$$

weights $\rightarrow \dots \subset \gamma_x^0 \subset \gamma_x^{-1} \subset \gamma_x^{-2} \dots$

so that the osculation of an element in the
flag belongs to the next element in the flag

(the osculation or horizontality condition $(\gamma_x^{(i)})^1 \subset \gamma_x^{(i-1)}$ $(**)$)

This is a curve in a fixed homogeneous space $SP(\Delta\mathfrak{g})/P$

where P is a parabolic subgroup of the symplectic group

W so that the curve is tangent to the natural
distribution in $SP(\Delta\mathfrak{g})/P$ ($\Leftrightarrow$ the osculation or horizontality
condition $(**)$)

§5 Extrinsic geometry of curves in flag varieties

In this way we reduced the original problem on
the intrinsic geometry of the distribution D to the extrinsic geometry
(w.r.t. the action of the (conformal) symplectic group) of the Jacobi curves
of its abnormal extremals. In particular, any extrinsic invariant
of the Jacobi curves is automatically an intrinsic invariant of the
original distribution. In particular for rank 2 distributions

the extended Jacobi curve is the curve of complex flags obtained by osculation from
the curve of the one-dimensional subspaces in this flag, i.e. from the curve in
projective space (a self-dual one, i.e. equivalent to its dual) and the classical
Wilczynski invariants of this curves give invariants of the original distributions

(In particular, for a rank 2 distribution on 5-dim manifold with s.g.v. $(2,3,5)$
the only nontrivial Wilczynski invariant of the Jacobi curve coincides with the Cartan biquadratic
form of 1910 "five variables" paper).

Similar to the Tanaka theory for intrinsic geometry
of filtered structures, one can construct the Tanaka-
like theory for such curves in the flag varieties with
similar main ingredients:

1) Pass from the filtered space $(*)$ to the

1) Pass from the filtered space $(*)$ to the associated graded space

$$V(\lambda) := \bigoplus \mathfrak{g}_\lambda^i(\lambda) / \mathfrak{g}_\lambda^{i+1}(\lambda) \quad (**)$$

2) Define the symbol of the curve $\text{Osc}(Y_\lambda)$ as the orbit S_λ of the tangent line

$$T_\lambda \text{Osc}(Y_\lambda) \in \text{csp}_{-1}(V(\lambda)),$$

upto the Ad-action of $\text{CSP}_0(V(\lambda))$
the space of degree -1 elements of $\text{csp}(V(\lambda))$

S_λ is called the Jacobi symbol (of D) at $\lambda \in \mathbb{P}W_D$

For fixed rank D and $\dim M$ the set of all possible Jacobi symbols is finite and classified (B. Doubrov and I.Z., 2012).

Note that finiteness is a general consequence of Vinberg 1976 on finiteness of the set of orbits of nilpotent elements in semisimple Lie algebra.

Finiteness and algebraic dependence of the extended Jacobi curve on the fibres of $\mathbb{P}W_D^1$ implies

that $\forall q \in M \exists$ a nonempty Zariski open subset U of the fiber $\mathbb{P}W_D$ s.t. the Jacobi symbols S_λ are the same for all $\lambda \in U$. This symbol is called the Jacobi symbol of distribution D at q . Finiteness of Jacobi symbols implies that the Jacobi symbol is constant in a neighborhood of a generic point on M .

Maximality of class condition

The distribution D is called of maximal class at $q \in M$.

if $\exists \lambda \in \mathbb{P}W_D$ and $i < 0$ s.t.

$$\mathfrak{g}_\lambda^i(\lambda) = \Delta_\lambda(\lambda) \quad \left(\text{the analog of the bracket-generating condition} \right)$$

Generic forms of distributions are of maximal class.

We are not aware of any rank 2 distribution with the small growth vector (s.g.v.) starting with $(2, 3, 5, \dots)$ or rank 3 distribution with s.g.v. starting with $(3, 6, \dots)$, which are not of maximal class at generic points.

Conjecture No such distributions exist.

From now on let $V := V(\lambda)$

3) Define the flat curve F_S of symplectic flags with given symbol S as

given symbol $\delta \in \mathcal{S}$

$\{ \exp^{x\delta} \text{ (fixed symplectic flag)}, x \in \mathbb{R} \}$,

i.e. the orbit of a fixed symplectic flag (with the appropriated dimensions of the spaces in it) w.r.t. to the action of the one-parametric subgroup generated by symbol δ .

5) By analogy with the Tanaka theory for the intrinsic geometry of filtered structures the algebra $\text{Sym}(F_\delta)$ of the flat curve with symbol δ is given by

$\text{Sym}(F_\delta) =$ the maximal subalgebra of $\text{osp}(V)$ s.t. its reductive part coincides with δ , i.e.

$$\text{Sym}(F_\delta)_- = \delta$$

6) Tanaka like theorem on frames:

To any curve of symplectic flags F with symbol δ one can assign the canonical bundle of moving frames of dimension equal to

$\dim \text{Sym}(F_\delta)$ (by a bundle of moving frames for a

a curve F we mean a subbundle (not necessary principle) of

the bundle $\Pi^{-1}(F) \rightarrow F$, where $\Pi: \text{OSP}(V) \rightarrow \text{OSP}(V)/\text{OSP}_{\geq 0}(V)$

is the canonical projection) (see reference [5] of the abstract for details)

§6 From extrinsic geometry of curves in symplectic flag variety to the intrinsic geometry of distributions.

Collecting all canonical frames in the fiber $\Delta(x)$ of the contact distribution for each x on the manifold N

we obtain the bundle with fibers of dimension =

$\dim \text{Sym}(F_\delta)$. This is not a principle $\text{Sym}(F_\delta)$

bundle but so-called quasi-principle bundle of type

$(\eta, \text{Sym}(F_\delta))$ where η is $\dim N$ -dimensional Heisenberg

algebra, the Tanaka symbol of Δ (see Example 16 in Sec 2 above)

The main theorem on absolute parallelism with given Jacobi symbol.
Theorem (B. Doubrov - I. Z., 2016)

1) To any distribution D of maximal class with constant Jacobi

1) To any distribution D of maximal class with constant Jacobi symbol S one can assign the canonical structure of absolute parallelism on the bundle over W_D of dimension $= \dim(\eta \oplus \text{symm}(F_S)) (= \text{the maximal nondegenerate } \mathbb{Z}\text{-graded Lie algebra having } \eta \oplus \text{symm}(F_S) \text{ as nonpositive part})$

$\swarrow$ negative part $\searrow$ degree 0 part

2) There exists the unique, up to a local diffeomorphism, distribution D_S with Jacobi symbol S having the symmetry group of dimension $= \dim \mathfrak{u}(\eta \oplus \text{symm}(F_S))$

Such unique distribution is called the symplectically flat distribution with Jacobi symbol S .

Example 1 $\text{rank } D = 2$, s.g.v. $= (2, 3, 5, \dots)$, $\dim M = n$

Rem Note that the condition that $\dim D^2 = 5$ is not restrictive: Indeed, if $\dim D^2 = 4$, then one can quotient M by the foliation of integral curves of the Cauchy characteristics I of D^2 & get the rank 2 distribution D^2/I on an $(n-1)$ -dim manifold.

If the s.g.v. of this new rank 2 distribution is $(2, 3, 5, \dots)$ then the rest can be applied to it (and the original distribution can be uniquely recovered from the reduced one). If the s.g.v. is still $(2, 3, 4, \dots)$ we repeat this step of factorization until either we get $(2, 3, 5, \dots)$ or we arrive to the contact $(2, 3)$ distribution, which would mean that the original distribution is Goursat and therefore has trivial geometry at a generic point (i.e. is isomorphic to the standard distribution in the jet space $J^{n-2}(\mathbb{R}, \mathbb{R})$ of

$(n-2)$ -jets of functions from $\mathbb{R}$ to $\mathbb{R}$ $\square$ -

• In this case

$$\text{rank } D = 2(n-3)$$

$$\text{Indeed } \dim (D^2)^\perp = \dim T^*M - 3 = 2n-3 \Rightarrow \dim \mathbb{P}(D^2)^\perp = 2n-4 \Rightarrow \dim \mathcal{N} = 2n-5 \Rightarrow$$

$\xrightarrow{\text{rank } D^2}$

$$\text{rank } \Delta = 2n-6;$$

• There is exactly one (for given n) Jacobi symbol generated by one nilpotent Jacobi block (of size $2(n-3) \times 2(n-3)$):

$$S = \text{span} \left\{ \begin{pmatrix} 0 & \dots & 0 \\ 1 & & \\ & \ddots & \\ 0 & \dots & 0 \\ & & 1 & 0 \end{pmatrix} \right\}$$

It is also convenient to encode the generator of S using one row diagram of length $2(n-3)$.



so that the boxes are filled by elements of a symplectic (Darboux) basis of V and S acts as the right shift on this basis;

• The flat curve is the curve of complex flags in V which is obtained by the osculation of the rational normal curve

$$t \mapsto [1 : t : t^2 : \dots : t^{2n-3}];$$

• $\text{symm}(F_S)$ is the irreducible embedding of $\mathfrak{gl}_2(\mathbb{R})$ into $\mathfrak{osp}(2n-6)$;

- The Tanaka prolongation of $\eta \oplus \text{symm}(F_3)$:

$$u(\eta \oplus \text{symm}(F_3)) = \begin{cases} \mathfrak{g}_2, & \text{if } n=5 \\ \eta \oplus \text{symm}(F_3) \cong \mathfrak{gl}_2(\mathbb{R}) \ltimes \eta, & \text{if } n>5 \end{cases}$$

semidirect
product

- The symplectically flat distribution D_g is the left-invariant distribution on the Lie group with the Lie algebra

$$\text{span}\{H, E_1, \dots, E_{n-2}, Z\}$$

having the following nontrivial brackets

$$[H, E_i] = E_{i+1} \quad 1 \leq i \leq n-3, \quad [E_1, E_2] = Z$$

$$\text{and such that } D_g(e) = \text{span}\{H, E_1\}$$

the group
identity

In particular, the small growth vector is equal to $(2, 5, 6, 7, \dots, n)$

increasing by 1 here

and for $n=6$ D_g corresponds to the flat distribution with the most degenerate Tanaka symbol, i.e. when the symmetric form $B: D \times D \rightarrow D^\perp$ is degenerate (see Lecture 1, §3, Example 1 b) above)

In fact the distribution D_g is locally equivalent to the natural rank 2 distribution associated with the Monge equation

$$z' = (y^{(n-3)})^2,$$

an underdetermined differential equation for 2 functions $y(x)$ and $z(x)$.

In more detail this is a rank 2 distribution on $\mathbb{R}^n$ with coordinates $(x, y, \dots, y^{(n-3)}, z)$ given by the following Pfaffian system (involving $n-2$ equations):

$$\begin{cases} dy^{(i)} - y^{(i+1)} dx = 0 & 0 \leq i \leq n-4 \\ dz - (y^{(n-3)})^2 dx = 0 \end{cases}$$

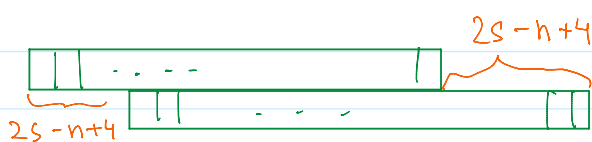
Note that for $n=5$ we get the Cartan-Hilbert equation

$$z' = (y^{(1)})^2$$

Example 2 rank $D=3$, s.g.v. is $(3, 6)$, $\dim M=n$
(very briefly)

Jacobi symbols correspond to two Jordan blocks of size $(n-3) \times (n-3)$ each encoded by a skew Young diagram with two rows of length $n-3$ each, depending on the additional integer parameter s with

$$\frac{n-4}{2} \leq s \leq n-5$$



$\hookrightarrow s$ controls the shift between rows

Subcase 1 If $s > \frac{n-4}{2}$ (the case of a nonrectangular diagram).

Let RV^m be the rational normal curve in r -dimensional projective space $\mathbb{P}^r$. Given a projective variety V in $\mathbb{P}^r$ let $T^b(V)$

projective space $\mathbb{P}^r$. Given a projective variety V in $\mathbb{P}^r$ let $T^b(V)$ be the b th tangential developable of V and let $S^b(V)$ be the b th secant variety of V , which is defined as the Zariski closure of the union of $(b-1)$ -planes, in $\mathbb{P}^r$ passing through b points of V . In this definition we set $S^0 V = V$, $S^1 V = V$.

Further, by $I(V)$ we denote the ideal of homogeneous polynomials (in the ambient vector space) vanishing on V and let $I_b(V)$ be the subspace of all polynomials of degree b in $I(V)$.

Then the degree k th component $y_k(\eta, \text{sym}(F_s))$ of the $\mathbb{Z}$ -graded Lie algebra $u(\eta \oplus \text{sym}(F_s))$ can be identified with

$$y_k(\eta, \text{sym}(F_s)) = I_{k+2}(S^k(T^{n-s-5}(RN^{(n-4)})))$$

= the space of homogeneous polynomials of degree $k+2$ vanishing on the k th secant variety of $(n-s-5)$ th tangential developable of the rational normal curve in $\mathbb{P}^{n-4}$.

In particular in the case when $s = n-5$ (the maximal shift in the skew Young diagram) o

$$y_k(\eta, \text{sym}(F_s)) = I_{k+2}(S^k(RN^{(n-4)})) \text{ and}$$

this space can be described explicitly by the space of the maximal minors of the catalectic (or Hankel) matrix

$$\begin{matrix}
 & & n-4-k \\
 & & \begin{matrix} k \rightarrow n-4 \\ r \rightarrow k \end{matrix} \\
 \begin{matrix} k+2 \\ \left(\begin{array}{cccc} 0!x_0 & 1!x_1 & 2!x_2 & \dots & (n-5-k)!x_{n-5-k} \\ 1!x_1 & 2!x_2 & 3!x_3 & \dots & (n-4-k)!x_{n-4-k} \\ \vdots & \vdots & \vdots & & \vdots \\ (k+1)!x_{k+1} & (k+2)!x_{k+2} & (k+3)!x_{k+3} & \dots & (n-4)!x_{n-4} \end{array} \right) \end{matrix}
 \end{matrix}$$

$$\dim I_{k+2}(S^k(RN^{(n-4)})) = \# \text{ of the } \begin{matrix} \text{maximal} \\ \text{minors} \end{matrix} = \binom{n-4-k}{k+2}$$

which will easily imply that

$$\dim(\text{sym } D_S) = \underbrace{\text{Fib}_{n-1}}_{\substack{\text{the dimension of} \\ \text{the symmetry group} \\ \text{of symplectically flat} \\ \text{distribution with Jacobi} \\ \text{symbol } S}} + n+2 \quad \text{-- exponential in } n$$

the dimension of
the symmetry group
of symplectically flat
distribution with Jacobi
symbol S

the $(n-1)$ st
Fibonacci
number, starting
with $\text{Fib}_1 = \text{Fib}_2 = 1$

Subcase 2 $s = \frac{n-4}{2}$ (the case of the rectangular Young diagram, possibly if n is even)

- if $n=6$ $u(\eta \oplus \text{sym}(F_S)) = \text{so}(3,4)$
- if n is even and greater than 6 $g_1(\eta, \text{sym } F_S) = 0 \Rightarrow$

$$u(\eta \oplus \text{sym}(F_S)) = \eta \oplus \text{sym}(S_S) = (\text{sl}_2(K) \times \text{gl}_2(K)) \ltimes \eta$$

where $\eta = \eta_{-1} \oplus \eta_{-2}$ with $\eta_{-1} = \underbrace{\text{Sym}^{n-4}(K^2)}_{\substack{\text{the } (n-3)\text{-dim} \\ \text{irreducible} \\ \text{sl}_2\text{-module}}} \otimes \underbrace{K^2}_{\substack{\text{the standard} \\ \text{gl}_2(K)\text{-module.}}}$

the $(n-3)$ -dim
irreducible
 sl_2 -module

the standard
 $\text{gl}_2(K)$ -module.