Theory of Besov spaces, Errotum

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Contents

1 Errortum

Legend Note: If the change is major, I will not mark the change by red.

$\star$ location

(a) I wrote.
(b) But I should have written.
(c) Dates
(d) Other comments if there are

$\star\star$ location

(a) Add
(b) What I want to add actually.
(c) Dates
(d) Other comments if there are

$\star\star\star$ location

(a) Remove
(b) What I want to remove actually.
(c) Dates
(d) Other comments if there are

1.1 Pages 1-9

1. Index 1. (p. xiii, Notation in This Book, Sets and Set Functions)
   (a) $\|x-y\|$
   (b) $|x-y|$
   (c) 29 Oct. 2019
2. Index 5. (p. xiii, Notation in This Book, Sets and Set Functions)
   (a) E is integrable over f
   (b) f is integrable over E
   (c) 29 Oct. 2019
3. Add as 14 (p. xiv, Notation in This Book, Sets and Set Functions)
   (a) For simplicity, for a measure μ, we write $\mu(\{\cdots\}) = \mu\{\cdots\}$.
   (b) 21 Aug. 2024
4. Add as 15 (p. xiv, Notation in This Book, Sets and Set Functions)
   (a) $S^{n-1} = \{x \in \mathbb{R}^n \,:\, |x| = 1\}$
   (b) 14 Jan. 2025
5. Index 18 (p. xiv, Notation in This Book, Function spaces)
   (a) $m_Q(f)$
   (b) $m_A(f)$
   (c) 5 Dec. 2023
6. Index 19. (p. xvii, Notation in This Book, Function spaces)
   (a) $\sup\limits_{Q \in \mathcal{D}_x} m_Q(|f|)$
   (b) $\sup\limits_{Q \in \mathcal{D}} m_Q(|f|)\chi_Q(x)$
   (c) 14 Jan. 2025
7. Index 27. (p. xvii, Notation in This Book, Function spaces)
   (a) The Kronecker delta function is given by
   $$\delta_{jk} \equiv \begin{cases} 1 & (j = k), \\ 0 & (j \neq k). \end{cases}$$
   for $j, k \in \mathbb{Z}$.
   (b) The Kronecker delta function defined on a set X is given by
   $$\delta_{jk} \equiv \begin{cases} 1 & (j = k), \\ 0 & (j \neq k) \end{cases}$$
   for $j, k \in X$.

(c) 29 Oct. 2019

(d) I should have expanded the range of the set on which δ is defined!

8. Index 28. (p. xvii, Notation in this Book)

(a) When two normed spaces X, Y are isomorphic, we write $X \approx Y$.

(b) When two normed spacess X, Y are isomorphic, we write $X \approx Y$.

(c) 15 Sep. 2020

1.2 Pages 10-19

1. **p. 3, line 3 from above:**
   (a) $\mathcal{N}$.
   (b) $\mathcal{N}$. Let λ_0 be the natural set function on the set of all measurable rectangles, generated by μ and ν.
   (c) 14 Sep. 2020
   (d) The set function λ_0 should have been defined.
2. **p. 3, Theorem 1.3:**
   (a) $\mu^*(E)$
   (b) $(\mu \otimes \nu)^*(E)$
   (c) 14 Sep. 2020
   (d) I should have defined $\mu \otimes \nu$ here.
3. **p. 4, Theorem 1.4:**
   (a) I should have combined "Theorem 1.4 3." with "Theorem 1.4 1.".
   (b) 15 Sep. 2020
4. **p. 5, line 4 from below, Theorem 1.5:**
   (a) $\mathrm{d}\lambda$.
   (b) $\mathrm{d}\lambda$. It will be understood that $\Phi(\infty) = \infty$.
   (c) 18 Sep. 2020
5. **p. 5, line 1 from below, the proof of Theorem 1.5:**
   (a) (λ, ∞)
   (b) $(\lambda, \infty]$
   (c) 18 Sep. 2020
6. **p. 6, lines 4 and 5 from above, the proof of Theorem 1.5:**
   (a) (λ, ∞)
   (b) $(\lambda, \infty]$
   (c) 18 Sep. 2020
7. **p. 6, line 5 from below:**
   (a) counter
   (b) counting
   (c) 15 Sep. 2020

1.3 Pages 10-19

1. **p. 13, line 8 from above, Definition 1.1:**
   (a) $\mathcal{D} = \mathcal{D}$
   (b) $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$
   (c) 14 Jan. 2025

2. **p. 13, lines 14 and 16 from above, Definition 1.2:**
   (a) dyadic cubes
   (b) dyadic cube
   (c) 14 Jan. 2025

3. **p. 14, line 16 from below, the proof of Theorem 1.7:**
   (a) for $j > J$
   (b) for $j > J + 1$
   (c) 14 Sep. 2020

4. **p. 14, line 13 from below, the proof of Theorem 1.7:**
   (a) $\mathbb{N} \to \{1, 2, \dots, n\}$
   (b) $\{1, 2, \dots, J_0 + 1\} \to \{1, 2, \dots, K\}$
   (c) 14 Sep. 2020

5. **p. 14, line 2 from below, the proof of Theorem 1.7:**
   (a) $\{J + 1, J + 2, \dots, J_0 + 1\}$
   (b) $\{1, J + 2, J + 3, \dots, J_0 + 1\}$
   (c) 14 Sep. 2020
   (d) Be noted that we need to consider numbers 1, $J + 2$, $J + 3$, ..., $J_0 + 1$.

6. **p. 14, line 2 from below, the proof of Theorem 1.7:**
   (a) $\Lambda_0 \to \Lambda$
   (b) $\Lambda \to \Lambda_0$
   (c) 14 Sep. 2020

7. **p. 19, line 4 from below:**
   (a) in Chapter 6.
   (b) in Chapter 6. We do not use §1.1.3 and the Choque integral except these places.
   (c) 14 Sep. 2020
   (d) I should have stressed that the Hausdorff capacity and the Choquet integral are not used till the end of the book.

1.4 Pages 20-29

1. **p. 20, line 10 from above:**
   (a) $H^0(E) \equiv \inf \left\{ N \in \mathbb{N} : E \subset \bigcup_{j=1}^{N} B(x_j, r_j) \right\}$,
   (b) $H^0(E) \equiv \inf \left(\left\{ N \in \mathbb{N} : E \subset \bigcup_{j=1}^{N} B(x_j, r_j) \right\} \cup \{\infty\} \right)$,
   (c) 14 Jan. 2025
2. **p. 20, line 12 from above:**
   (a) of open balls
   (b) of bounded open balls
   (c) 14 Jan. 2025
3. **p. 22, line 1 from below, Example 1.2:**
   (a) $m \in \mathbb{Z}$
   (b) $m \in \mathbb{Z}^2$
   (c) 14 Sep. 2020
4. **p. 23, the beginning of the proof of Proposition 1.6:**
   (a) Let
   (b) Let $E \subset \mathbb{R}^n$. Let
   (c) 14 Sep. 2020
   (d) I should have declared that E is a subset of $\mathbb{R}^n$.
5. **p. 25, line 8 from below, the proof of Theorem 1.10:**
   (a) and
   $$\tilde{H}_0^d(E_1 \cup E_2)$$
   (b) 29 Oct. 2019
   (c) and
   $$\tilde{H}_0^d(E_1 \cap E_2)$$
   (d) 29 Oct. 2019
6. **p. 25, (1.32), the proof of Theorem 1.11:**
   (a) Remove
   (b)
   $$K \equiv \bigcap_{j=1}^{\infty} K_j$$
   (c) 14 Sep. 2020

(d) K is already defined in the statement of Theorem 1.11.

7. **p. 26, line 2 from above, the proof of Theorem 1.11:**

   (a) Thus,

   (b) Since $\{K_j\}_{j=1}^{\infty}$ is decreasing,

   (c) 10 Dec. 2024

8. **p. 26, line 9 from below, the proof of Lemma 1.5:**

   (a) and

   (b) for each $j \in \mathbb{N}$ and

   (c) 14 Jan. 2025

9. **p. 27, line 3 from above, the proof of Lemma 1.5:**

   (a) $O_m \cap \mathrm{Int}$

   (b) $O_m \cap$

   (c) 14 Jan. 2025

10. **p. 28, line 9 from below, the proof of Lemma 1.5:**

    (a) $\{Q_{2,k}^{*}\}_{k \in J_1}$

    (b) $\{Q_{2,k,m}^{*}\}_{k \in J}$

    (c) 14 Jan. 2025

11. **p. 28, line 5 from below, the proof of Lemma 1.5:**

    (a) a collection $\{Q_{m,k}^{(j)}\}_{k \in I_{j,m}}$ for

    (b) collections $\{Q_{m,k}^{(j)}\}_{k \in I_{j,m}}$ and $\{Q_{j,k,m}^{*}\}_{k \in J_j}$ for

    (c) 14 Jan. 2025

1.5 Pages 30-39

1. **p. 30:**

   (a) one line above Definition 1.6

   (b) Definition 1.6

   (c) Example 1.3

   (d) line 2 from below, Definition 1.7

   (a) (λ, ∞)

   (b) $(\lambda, \infty])$

   (c) 14 Sep. 2020

   (d) $f^{-1}(\lambda, \infty) = \{x \in \mathbb{R}^n : \lambda < f(x) < \infty\}$, while $f^{-1}(\lambda, \infty] = \{x \in \mathbb{R}^n : \lambda < f(x)\}$,

2. **p. 31:**

   (a) line 10 from above, the proof of Proposition 1.7

   (b) line 11 from above, the proof of Proposition 1.7

   (c) line 1 from below, Lemma 1.6

   (a) (λ, ∞)

   (b) $(\lambda, \infty])$

   (c) 18 Sep. 2020

   (d) $f^{-1}(\lambda, \infty) = \{x \in \mathbb{R}^n : \lambda < f(x) < \infty\}$, while $f^{-1}(\lambda, \infty] = \{x \in \mathbb{R}^n : \lambda < f(x)\}$,

3. **p. 32:**

   (a) line 4 from above, the proof of Lemma 1.6

   (b) line 5 from above (three times), the proof of Lemma 1.6

   (c) line 6 from above, the proof of Lemma 1.6

   (d) line 7 from above, the proof of Lemma 1.6

   (a) (λ, ∞)

   (b) $(\lambda, \infty])$

   (c) 14 Sep. 2020

   (d) $f^{-1}(\lambda, \infty) = \{x \in \mathbb{R}^n : \lambda < f(x) < \infty\}$, while $f^{-1}(\lambda, \infty] = \{x \in \mathbb{R}^n : \lambda < f(x)\}$,

4. **p. 34, line 7 from above, the proof of Lemma 1.8:**

   (a) $\geq$

   (b) $=$

   (c) 14 Sep. 2020

   (d) This is nothing but the definition of the second term in the line in question.

1.6 Pages 40-49

1.7 Pages 50-59

1.8 Pages 60-69

1. **p. 62, Theorem 1.27:**
   (a) However,
   (b) However, including the fact that $\partial^\alpha(f * h)$ is continuous for each $\alpha \in \mathbb{N}_0{}^n$,
   (c) 29 May 2021
   (d) In part 2, I should have proved that $\partial^\alpha(f * h)$ is continuous for each $\alpha \in \mathbb{N}_0{}^n$.

2. **p. 63, line 7 from below, Definition 1.19 2:**
   (a) $\psi_j * f$
   (b) $\psi_j(D)f$
   (c) 18 Oct, 2022

3. **p. 67, (1.97):**
   (a) Remove
   (b) $\frac{1}{(2\pi)^{\frac{n}{2}}}$
   (c) 4 Jan, 2021

4. **p. 67, (1.98):**
   (a) Remove
   (b) $\frac{1}{\sqrt{(2\pi)^n}}$
   (c) 4 Jan, 2021

5. **p. 68, (1.99):**
   (a) $\int$
   (b) $\int_{\mathbb{R}^n}$
   (c) 14 Feb, 2022

1.9 Pages 70-79

1.10 Pages 80-89

1.11 Pages 90-99

1.12 Pages 100-109

1. **p. 101, Proposition 1.15:**

   (a) $\sup_{y \in B(x,2^{-j})} \left| f^j(y) - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le d-1} \frac{1}{\alpha!} \frac{\partial^\alpha f^j}{\partial x}(x)(y-x)^\alpha \right|$

   (b) $m^{(u)}_{B(x,2^{-j})} \left(f - \sum_{\alpha \in \mathbb{N}_0{}^n, |\alpha| \le d-1} \frac{\partial^\alpha f^j}{\partial x^{\alpha}}(x)(\star - x)^\alpha \right)$

   (c) 24 April, 2024

2. **p. 101, line 1 from below, (1.131), the proof of Proposition 1.15:**

   (a) $\frac{\partial^\alpha f^j}{\partial x}$

   (b) $\frac{\partial^\alpha f^j}{\partial x^{\alpha}}$

   (c) 14 Jan. 2025

3. **p. 102, line 7 from above, the proof of Proposition 1.15:**

   (a) $\frac{\partial^\alpha f^j}{\partial x}$

   (b) $\frac{\partial^\alpha f^j}{\partial x^{\alpha}}$

   (c) 14 Jan. 2025

1.13 Pages 110-119

1. **p. 111, line 8 from below, the proof of Proposition 1.17:**
   (a) can define
   (b) can select
   (c) 14 Jan. 2025

2. **p. 112, line 9 from above, the proof of Proposition 1.17:**
   (a) the proof is complete.
   (b) the proof of the first inequality is complete. The second inequality can be proved by reexamining the proof; see the last two steps of this proof.
   (c) 27 Nov. 2024

3. **p. 112:**
   - line 6 from below, the proof of Corollary 1.3
   - line 5 from below, the proof of Corollary 1.3

   (a) (λ, ∞)
   (b) $(\lambda, \infty])$
   (c) 18 Sep. 2020
   (d) $\chi_{(\lambda,\infty)}(|f|) = \chi_{\{x\in\mathbb{R}^n:\lambda<|f(x)|<\infty\}}$, while $\chi_{(\lambda,\infty]}(|f|) = \chi_{\{x\in\mathbb{R}^n:\lambda<|f(x)|\}}$

4. **p. 113, line 2 from above, the proof of Corollary 1.3:**
   (a) $(2\lambda, \infty)$
   (b) $(2\lambda, \infty])$
   (c) 18 Sep. 2020
   (d) $\chi_{(2\lambda,\infty)}(M(x)) = \begin{cases} 1 & (2\lambda < M(x) < \infty) \\ 0 & \text{otherwise} \end{cases}$, while $\chi_{(2\lambda,\infty]}(M(x)) = \begin{cases} 1 & (2\lambda < M(x)) \\ 0 & \text{otherwise} \end{cases}$

5. **p. 112:**
   - line 3 from above, the proof of Corollary 1.3
   - line 6 from above, the proof of Corollary 1.3
   - line 7 from above, the proof of Corollary 1.3

   (a) (λ, ∞)
   (b) $(\lambda, \infty])$
   (c) 18 Sep. 2020
   (d) $\chi_{(\lambda,\infty)}(|f(x)|) = \begin{cases} 1 & (\lambda < |f(x)| < \infty) \\ 0 & \text{otherwise} \end{cases}$, while $\chi_{(\lambda,\infty]}(|f(x)|) = \begin{cases} 1 & (\lambda < |f(x)|) \\ 0 & \text{otherwise} \end{cases}$

1.14 Pages 120 - 129

1. **p. 120, Definition 1.35:**
   (a) and let $J \subset \mathbb{Z}$
   (b) and let J a countable set
   (c) 14 Jan. 2025
2. **p. 120, Definition 1.35:**
   (a) $\{f_j\}_{j=1}^{\infty}$
   (b) $\{f_j\}_{j \in J}$
   (c) 15 Sep. 2020
3. **p. 120, Lemma 1.22:**
   (a) Let $1 \le p, q \le \infty$.
   (b) Let $1 \le p, q < \infty$.
   (c) 29 Oct. 2019
   (d) Clearly there is a counterexample if $p + q = \infty$. The proof of Theorem 1.49 uses Lemma 1.22 but this modified version of Lemma 1.22 can be used there.
4. **p. 121, line 8 from above:**
   (a) $\left(\sum_{j=1}^{\infty} |f_j(x)|^q\right)^{\frac{p}{q}}$
   (b) $A^{-p'}\left(\sum_{j=1}^{\infty} |f_j(x)|^q\right)^{\frac{p}{q}}$
   (c) 20 Oct. 2021
5. **p. 127, line 6 from above:**
   (a) $|\varphi(u)|^{\eta}$
   (b) $|\varphi(x-u)|^{\eta}$
   (c) 20 Oct. 2021
6. **p. 128, Theorem 1.50 and Corollaries 1.5 and 1.6:**
   (a) I should have alluded to the fact that we can relax the assumption $f \in \mathcal{S}_{B(1)}(\mathbb{R}^n)$ to $f \in \mathcal{S}'_{B(1)}(\mathbb{R}^n)$ by reexamining the proof of Proposition 1.19. This will be used in Chapter 6.
   (b) 2023 Jan. 31

1.15 Pages 130 - 139

1. **p. 135, line 2 from above, the proof of Theorem 1.54:**
   (a) $B(2^j t) \setminus B(t)$
   (b) $B(2^j t) \setminus B(2^{j-1} t)$
   (c) 14 Jan. 2025
2. **p. 135, Theorem 1.55:**
   (a) $\eta \perp \mathcal{P}_{N-1}$ satisfies the differential inequality
   (b) $\eta \in \mathcal{P}_{N-1}(\mathbb{R}^n)$ satisfies the inequality
   (c) 15 Sep. 2020
   (d) The verb duplicates.
3. **p. 137, lines 8 and 1 from below, the proof of Theorem 1.55:**
   (a) $\mathbb{R}^n \setminus B(t)$
   (b) $\mathbb{R}^n \setminus B(1)$
   (c) 14 Jan. 2025

1.16 Pages 140-149

1. **p. 144, one line below (1.180), line 9 from below, the proof of Theorem 1.57:**
   (a) Then O is an open set and O satisfies
   (b) Then O_λ is an open set and O_λ satisfies
   (c) 23 Dec. 2019
2. **p. 144, three lines below (1.180), line 11 from below, the proof of Theorem 1.57:**
   (a) We see from the definition of $\widehat{O_\lambda}$ $B(y,t) \subset O$
   (b) From the definition of $\widehat{O_\lambda}$ we see $B(y,t) \subset O_\lambda$
   (c) 23 Dec. 2019
3. **p. 148, line two from below, Theorem 1.59:**
   (a) $L^\infty(\mathbb{R}^n)$-bounded
   (b) belongs to $L^\infty(\mathbb{R}^n)$
   (c) 14 Feb. 2022
4. **p. 149, line six from above, Theorem 1.59:**
   (a) $\{Q_j\}_j$
   (b) $\{Q_j\}_{j \in J}$
   (c) 14 Feb. 2022
5. **p. 149, line 11 from below, the proof of Theorem 1.59:**
   (a) nonoverlapping
   (b) nonoverlapping in the sense that $\mathrm{supp}(b_j)$ and $\mathrm{supp}(b_{j'})$ do not have a common interior point if j and j' are distinct.
   (c) 25 May, 2021
6. **p. 149, line 10 from below, the proof of Theorem 1.59:**
   (a) moment condition
   (b) L^1 condition
   (c) 25 May, 2021

1.17 Pages 150-159

1. **p. 156, Definition 1.42:**
   (a) $x \in \mathbb{R}^n$
   (b) $x \in \mathbb{R}^n \setminus \{0\}$
   (c) 26 May 2021

2. **p. 158, Theorem 1.62:**
   (a) The implicit constant in the inequality in the conclusion can be chosen as a constant multiple of the sum of $\|T\|_{L^2 \to L^2}$ and the two implicit constants in (1.214) and (1.215)
   (b) 4 Jun. 2021
   (c) Reference: Used in the proof of Theorem 3.2 on p. 325

3. **p. 159, Theorem 1.63:**
   (a) conditions:
   (b) conditions: For all $f \in L^p(\mathbb{R}^n)$, $Sf \geq 0$ a.e.
   (c) 28 Oct. 2020

4. **p. 159, line 5 from below, Theorem 1.63:**
   (a) $|Sf|$
   (b) Sf
   (c) 28 Oct. 2020

5. **p. 159, line 2 from below, Theorem 1.63:**
   (a) $f \in L^p(\mathbb{R}^n)$
   (b) $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$
   (c) 28 Oct. 2020

1.18 Pages 160-169

1. **p. 160, line 1 from above:**
   (a) $|Sf|$
   (b) Sf
   (c) Date: 28 Oct. 2020
2. **p. 160, line 4 from above:**
   (a) $|Sf|$
   (b) Sf
   (c) Date: 28 Oct. 2020
3. **p. 160, line 4 from above:**
   (a) $|S[\chi_{(0,\lambda]}(|f|)f]|$
   (b) $S[\chi_{(0,\lambda]}(|f|)f]$
   (c) Date: 28 Oct. 2020
4. **p. 160, line 8 from above:**
   (a) $\displaystyle\int_{\mathbb{R}^n}$
   (b) $\displaystyle\int_{\mathbb{R}^n} \lambda^{q-1}$
   (c) Date: 28 Oct. 2020
5. **p. 160, lines 4, 6, and 8 from above:**
   (a) (λ, ∞)
   (b) $(\lambda, \infty]$
   (c) Date: 18 Sep. 2020
   (d) See the remark on the erratum of p. 30.
6. **p. 160, Theorem 1.64:**
   (a) conditions:
   (b) conditions: For all $f \in L^p(\mathbb{R}^n)$, $Sf \geq 0$ a.e.
   (c) 28 Oct. 2020

1.19 Pages 170-179

1.20 Pages 180-189

1.21 Pages 190-199

1.22 Pages 200-209

1.23 Pages 210-219

1.24 Pages 220-229

1. **p. 227, Line 7, the proof of Theorem 2.7:**
   (a) $|y|$
   (b) $2^j|y|$
   (c) Date: 10 Dec. 2021

1.25 Pages 230-239

1.26 Pages 240-249

1.27 Pages 250-259

1.28 Pages 260-269

1.29 Pages 270-279

1. **p. 278, Exercise 250:**

   (a) $\mathrm{d}x$

   (b) $\mathrm{d}x \quad (f \in \mathcal{S}(\mathbb{R}^n))$

   (c) Date: 14 Sep. 2020

   (d) The domain of ℓ_α is $\mathcal{S}(\mathbb{R}^n)$.

2. **p. 278, Exercise 250:**

   (a) $\mathcal{S}(\mathbb{R}^n) = \bigcap_{\alpha \in \mathbb{N}_0{}^n} \ker(\ell_\alpha)$

   (b) $\mathcal{S}_\infty(\mathbb{R}^n) = \bigcap_{\alpha \in \mathbb{N}_0{}^n} \ker(\ell_\alpha)$

   (c) Date: 14 Sep. 2020

1.30 Pages 280-289

1.31 Pages 290-299

1. **p. 292, line 7 from above, the proof of Proposition 214:**
   (a) $L \geq 2[A+1]$,
   (b) $N \geq 2[A+1]$,
   (c) Date: Sept. 24 2021
2. **p. 292, line 9 from above, p. 292, line 3 from below, p. 292, line 1 from below, p. 293, line 2 from above, p. 293, line 4 from above, the proof of Proposition 214:**
   (a) $(j-l)(L+1)$
   (b) $(j-l)(A+1)$
   (c) Date: Sept. 24 2021
3. **p. 293, line 6 from above, p. 293, line 9 from above, p. 293, line 12 from above, the proof of Proposition 214:**
   (a) $(j-l)(L+1-A+n)$
   (b) $(j-l)(N+1-A+n)$
   (c) Date: Sept. 24 2021
4. **p. 295, the first line of the proof of Theorem 2.34:**
   (a) We prove *1* since we can prove *2* similarly.
   (b) We prove *2* since we can prove *1* similarly.
   (c) Date: Sept. 24 2021
5. **p. 297, Theorem 2.35:**
   (a) $L > \sigma_p + s$
   (b) $L > s$
   (c) Date: Sept. 24 2021
6. **p. 297, Theorem 2.35:**
   (a) $L > \sigma_{pq} + s$
   (b) $L > s$
   (c) Date: Sept. 24 2021

1.32 Pages 300-309

1.33 Pages 310-319

1.34 Pages 320-329

1.35 Pages 330-339

1. **p. 333, Lemma 3.1:**
   (a) Let
   (b) Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (3.18).
2. **p. 334, Lemma 3.2:**
   (a) Let
   (b) Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (3.18).

1.36 Pages 340-349

1.37 Pages 350-359

1. **p. 357, line 18 from above, the proof of Proposition 3.5:**
   (a) $\langle f, g \rangle_{\mathcal{H}} = \dfrac{1}{I^{(j)}} \langle f, g \rangle$
   (b) $\langle f, g \rangle_{\mathcal{H}_j} = \dfrac{1}{I^{(j)}} \langle f, \varphi^{(j)} g \rangle$
   (c) 29 Oct. 2019, 15 Sep. 2020

2. **p. 358, line 1 from below, the proof of Lemma 3.8:**
   (a) If we combine
   (b) Remark that (3.67) is used to deduce (3.68). If we combine
   (c) 29 Oct. 2019

3. **p. 359, (3.73), the proof of Lemma 3.8:**
   (a) $=$
   (b) $\equiv$.
   (c) 29 Oct. 2019

4. **p. 359, (3.74), the proof of Lemma 3.8:**
   (a) $=$
   (b) $\equiv$.
   (c) 29 Oct. 2019

5. **p. 359, (3.73), the proof of Lemma 3.8:**
   (a) $=$
   (b) $\equiv$.
   (c) 29 Oct. 2019

6. **p. 359, line 3 from below, the proof of Lemma 3.8:**
   (a) $\leq$
   (b) $\lesssim$
   (c) 29 Oct. 2019

1.38 Pages 360 - 369

1. **p. 362, line 1 from below, the proof of Proposition 3.6:**
   (a) $\sup_{j \in \mathbb{N}}$
   (b) $\sup_{j \in \mathbb{N}, k \in K_j}$
   (c) 29 Oct. 2019
2. **p. 363, line 5 from above, the proof of Proposition 3.6:**
   (a) inequalities and (3.72)
   (b) inequalities
   (c) 29 Oct. 2019
3. **p. 368, line 5 from above, Proposition 3.8:**
   (a) $\sum_{j=1}^{\infty}$
   (b) $\sum_{l=1}^{\infty}$
   (c) 29 Oct. 2019
4. **p. 368, line 5 from above, Proposition 3.8:**
   (a) $\ell(Q_j)$
   (b) $\ell(Q_l)$
   (c) 29 Oct. 2019
5. **p. 368, line 9 from above:**
   (a) to partition
   (b) partition
   (c) 15 Sep. 2020

1.39 Pages 370 - 379

1. **p. 371, (3.93) (twice):**
   (a) Q_l
   (b) Q_j
   (c) 29 Oct. 2019
2. **p. 379, Exercise 3.34:**
   (a) Let $f \in H^p(\mathbb{R}^n)$.
   (b) Let $f \in H^p(\mathbb{R}^n)$ in the sense that $\mathcal{M}f \in L^p(\mathbb{R}^n)$.
   (c) 9 Nov. 2019

1.40 Pages 380 - 389

1.41 Pages 390 - 399

1.42 Pages 400 - 409

1. **p. 400, line 7 from above, the proof of Theorem 3.19:**
   (a) the continuous variable
   (b) the continuous variables
   (c) 14 Sep. 2020
2. **p. 400, 4 lines below (3.133), the proof of Theorem 3.19:**
   (a) there exists a proper dyadic cube with respect to Q
   (b) there exist a set Z of measure zero and $S \in \mathcal{D}(Q) \setminus \{Q\}$
   (c) 14 Sep. 2020
   (d) Recall that we need to use the Lebesgue differentiation theorem.
3. **p. 400, 4 lines below (3.133), the proof of Theorem 3.19:**
   (a) as long as $x \in Q$
   (b) as long as $x \in Q \setminus Z$
   (c) 14 Sep. 2020
   (d) Recall that we need to use the Lebesgue differentiation theorem.
4. **p. 402, line 2 from above, the proof of Corollary 3.5:**
   (a) $\mathrm{d}\lambda$
   (b) $\mathrm{d}\lambda \times |Q|$
   (c) 14 Jan. 2025
5. **p. 402, line 3 from above, the proof of Corollary 3.5:**
   (a) $\|b\|_{\mathrm{BMO}}{}^p$
   (b) $\|b\|_{\mathrm{BMO}}{}^p \times |Q|$
   (c) 14 Jan. 2025
6. **p. 402, line 4 from above, the proof of Corollary 3.5:**
   (a) $\|b\|_{\mathrm{BMO}}{}^p$
   (b) $\|b\|_{\mathrm{BMO}}{}^p \times |Q|$
   (c) 14 Jan. 2025

1.43 Pages 410 - 419

1.44 Pages 420 - 429

1.45 Pages 430 - 439

1. **p. 434, 4 lines below (4.8):**
   (a) over j
   (b) over ν
   (c) 12 Jan. 2022
2. **p. 435, line 2 from above:**
   (a) $\leq$
   (b) $\lesssim$
   (c) 12 Jan. 2022
   (d) I argue as follows:

$$
\begin{aligned}
&\left|\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\langle 2^\nu x-m\rangle^{-\kappa-\varepsilon}\right| \\
&\leq \sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\,\langle 2^\nu x-m\rangle^{-\kappa-\varepsilon} \\
&= \sum_{m\in\mathbb{Z}^n}\chi_{[-1,1]^n}(2^\nu x-m)\,|\lambda_{\nu m}|\,\langle 2^\nu x-m\rangle^{-\kappa-\varepsilon} \\
&\quad + \sum_{m\in\mathbb{Z}^n}\sum_{j=1}^{\infty}\chi_{(-2^j,2^j)^n\setminus(-2^{j-1},2^{j-1})^n}(2^\nu x-m)\,|\lambda_{\nu m}|\,\langle 2^\nu x-m\rangle^{-\kappa-\varepsilon} \\
&\leq \sum_{m\in\mathbb{Z}^n}\chi_{(-1,1)^n}(2^\nu x-m)\,|\lambda_{\nu m}| \qquad + \sum_{m\in\mathbb{Z}^n}\sum_{j=1}^{\infty}\chi_{(-2^j,2^j)^n\setminus(-2^{j-1}} \\
&\lesssim \sum_{j=0}^{\infty}\left(2^{-j(\kappa+\varepsilon)}\sum_{m\in\mathbb{Z}^n,|2^\nu x-m|_\infty<2^j}|\lambda_{\nu m}|\right)
\end{aligned}
$$

1.46 Pages 440 - 449

1. **p. 448, line 6 from above:**
   (a) δ_{jk}
   (b) δ_{k0}
   (c) 15 Sep. 2020

1.47 Pages 450-459

1.48 Pages 460-469

1.49 Pages 470-479

1. **p. 474, lines 6 and 5 from below:**

   (a) $x \in X_0 \cap X_1$

   (b) $x \in X_0 \cap X_1 \setminus \{0\}$

   (c) 14 Jan. 2025

1.50 Pages 480-489

1.51 Pages 490-499

1. **p. 497, (4.83), the proof of Theorem 4.31:**
   (a) $({X_0}^*, {X_1}^*)_\theta$
   (b) $[{X_0}^*, {X_1}^*]_\theta$
   (c) 12 Dec. 2024

2. **p. 497, one line below (4.83), the proof of Theorem 4.31:**
   (a) Therefore,
   (b) Assume that g assumes its value in $X_0 \cap X_1$ and that g, restricted to S, is an $X_0 \cap X_1$-valued holomorphic function. Since $x^* = f'(\theta)$ and $x = g(\theta)$,
   (c) 4 Jan. 2025

3. **p. 497, line 6 from below, the proof of Theorem 4.31:**
   (a) $\|g(z+h)\|_{(X_0 \cap X_1)^*}$
   (b) $\|g(z+h)\|_{X_0 \cap X_1}$
   (c) 4 Jan. 2025

4. **p. 497, line 2 from below, the proof of Theorem 4.31:**
   (a) $f^*(z)(g(z))$
   (b) $f^{*\prime}(z)(g(z))$
   (c) 12 Dec. 2024

5. **p. 498, line 1 from above, the proof of Theorem 4.31:**
   (a) $({X_0}^*, {X_1}^*)_\theta$
   (b) $[{X_0}^*, {X_1}^*]^\theta$
   (c) 12 Dec. 2024

6. **p. 498, line 2 from above, the proof of Theorem 4.31:**
   (a) $x \in [{X_0}^*, {X_1}^*]^\theta$
   (b) $x \in [{X_0}^*, {X_1}^*]^\theta$ and $\|x\|_{[X_0, X_1]_\theta{}^*} \le \|x\|_{[{X_0}^*, {X_1}^*]^\theta}$
   (c) 23 Sept. 2024

7. **p. 499, line 3 from above, the proof of Theorem 4.31:**
   (a) $k_a(\tau)$
   (b) $k_a(\mu(j + i\tau))$
   (c) 23 Sept. 2024

8. **p. 499, line 4 from above, the proof of Theorem 4.31:**
   (a) $\mathrm{Lip}(\mathbb{R}; X_0)$
   (b) $\mathrm{Lip}(\mathbb{R}, X_0^*)$

(c) 4 Jan. 2025

9. **p. 499, line 5 from above, the proof of Theorem 4.31:**

   (a) $\mathrm{Lip}(\mathbb{R}; X_1)$

   (b) $\mathrm{Lip}(\mathbb{R}, X_1^*)$

   (c) 4 Jan. 2025

10. **p. 499, line 5 from above, the proof of Theorem 4.31:**

    (a) $\tau \in \partial\Delta(1)$

    (b) $\tau \in \mathbb{R}$ and $j = 0, 1$

    (c) 23 Sept. 2024

11. **p. 499, line 7 from above, the proof of Theorem 4.31:**

    (a) We define

    (b) By using the density assumption we define

    (c) 23 Sept. 2024

12. **p. 499, line 10 from above, the proof of Theorem 4.31:**

    (a) $g(z) \equiv \displaystyle\int_{\frac{1}{2}}^{\mu^{-1}(z)} k(\zeta)\mathrm{d}\zeta$

    (b) $g(z) \equiv \displaystyle\int_{\mu^{-1}(\frac{1}{2})}^{z} k(\mu(\zeta))\mathrm{d}\zeta$

    (c) 18 Dec. 2024

13. **p. 499, line 10 from below, the proof of Theorem 4.31:**

    (a) $\|g_0\|_{L^\infty(X_0{}^*)}$

    (b) $\|g_0\|_{\mathrm{Lip}(\mathbb{R}, X_0{}^*)}$

    (c) 14 Jan. 2025

14. **p. 499, line 10 from below, the proof of Theorem 4.31:**

    (a) $\|g_1\|_{L^\infty(X_1{}^*)}$

    (b) $\|g_1\|_{\mathrm{Lip}(\mathbb{R}, X_1{}^*)}$

    (c) 14 Jan. 2025

15. **p. 499, line 9 from below, the proof of Theorem 308:**

    (a) $l(f(\theta))$

    (b) $x^*(f(\theta))$

    (c) 23 Sept. 2024

16. **p. 499, line 8 from below, the proof of Theorem 308:**

    (a) $-g(\theta)$

(b) $-ig(\theta)$

(c) 23 Sept. 2024

17. **p. 499, line 7 from below, the proof of Theorem 308:**

    (a) $g'(\theta)$

    (b) $g'(\theta)$

    (c) 23 Sept. 2024

    (d) The position of $'$ is strange.

18. **p. 499, line 6 from below, the proof of Theorem 308:**

    (a) $l = g'(\theta) \in [\mathcal{X}_0{}^*, \mathcal{X}_1{}^*]^\theta = [\mathcal{X}_0{}^*, \mathcal{X}_1{}^*]_\theta$

    (b) $x^* = g'(\theta) \in [\mathcal{X}_0{}^*, \mathcal{X}_1{}^*]^\theta$

    (c) 17 Dec. 2024

    (d) The position of $'$ is strange.

1.52 Pages 500-509

1. **p. 501, line 6 from above, the proof of Lemma 4.11**:
   (a) Define
   (b) For $t > 0$ define
   (c) 14 Jan. 2025

2. **p. 503, line 18 from above, the proof of Theorem 4.33**:
   (a) We assume $\|x\|_{[Y_0,Y_1]^\theta} \le 1$.
   (b) We assume $\|x\|_{[Y_0,Y_1]^\theta} < 1 - \varepsilon$ for some $\varepsilon > 0$.
   (c) 14 Jan. 2025

3. **p. 503, line 24 from above, the proof of Theorem 4.33**:
   (a) $y^*(\theta)(x)$
   (b) $y^*(x)$
   (c) 14 Jan. 2025

4. **p. 503, line 24 from above, the proof of Theorem 4.33**:
   (a) If we use the Poisson integral expression and the change of variables, then we obtain Theorem 4.33.
   (b) To conclude the proof, we need to show $|y^*(x)| \le 1$. To this end, we have only to prove $|g_j(z)| \le 1$ for all $z \in \bar{S}$. Instead of using g_j, we can consider $2^{j+k}(G^*(z + 2^{-j}i)(G(z+2^{-k}i) - G(z)) - G^*(z)(G(z+2^{-k}i) - G(z)))$ and we have only to show that $|2^{j+k}(G^*(z + 2^{-j}i)(G(z + 2^{-k}i) - G(z)) - G^*(z)(G(z + 2^{-k}i) - G(z)))| \le 1$ for all $z \in \bar{S}$. Thus, by the three-line lemma we may let $z \in \partial S$. In this case, using $\|G\|_{\mathcal{G}(\mathcal{X}_0,\mathcal{X}_1)} < 1 - \varepsilon$ and $\|G^*\|_{\mathcal{G}((\mathcal{Y}_0)^*,(\mathcal{Y}_1)^*)} < 1 + \varepsilon$, we obtain $|2^{j+k}(G^*(z + 2^{-j}i)(G(z+2^{-k}i) - G(z)) - G^*(z)(G(z+2^{-k}i) - G(z)))| \le 1 - \varepsilon^2$. Thus, we obtain Theorem 4.33.
   (c) 14 Jan. 2025

1.53 Pages 510-519

1.54 Pages 520-529

1. **p. 529, Exercise 4.51 1. and 2.:**

   (a) $\mathbb{R}^n_+$

   (b) $(-1,1)^n$

   (c) 14 Sep. 2020

   (d) Since $\mathbb{R}^n_+$ is unbounded, it is impossible that its indicator function belongs to $L^p(\mathbb{R}^n)$ for $p < \infty$. In addition, what is written in [110], is not about $\mathbb{R}^n_+$.

1.55 Pages 530-539

1.56 Pages 540-549

1.57 Pages 550-559

1.58 Pages 560-569

1.59 Pages 570-579

1. **p. 570, Proof of Theorem 5.3:**
   (a) Hestenes
   (b) Hestenes's
   (c) 29 Oct. 2019

1.60 Pages 580-589

1.61 Pages 590-599

1.62 Pages 600-609

1.63 Pages 610-619

1.64 Pages 620-629

1. **p. 624, Lemma 5.10:**
   (a) a sectorial operator on a Banach space X.
   (b) a sectorial operator on a Banach space X as in Definition 5.13, and let $0 < \eta < \mu < \pi$.
   (c) 29 Oct. 2019
   (d) I should have clarified the role of μ.
2. **p. 629, one line above Lemma 5.12:**
   (a) iv
   (b) iv
   (c) 15 Sep. 2020
3. **p. 629, the second line of the proof of Lemma 5.12:**
   (a) $\rangle_{N_1}$.
   (b) $\rangle_{N_1}$ for all $v \in \mathrm{Dom}(D)$.
   (c) 15 Sep. 2020

1.65 Pages 630-639

1. **p. 630, the second line of the proof of Proposition 5.11:**
   (a) are dense in H_0
   (b) are H_0.
   (c) 29 Oct. 2019
   (d) H_0 itself!
2. **p. 631, Theorem 5.22:**
   (a) operator.
   (b) operator. Assume that $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$ for some $0 < \theta^* < \dfrac{\pi}{2}$.
   (c) 29 Oct. 2019
3. **p. 632, line 4 from above, the proof of Theorem 5.22:**
   (a) $L_a u \to u$
   (b) $L_a u \to Lu$.
   (c) 29 Oct. 2019
4. **p. 632, line 6 from above, the proof of Theorem 5.22:**
   (a) $f(L_a)u$
   (b) $f(L_a)(\mathrm{id}_H + L_a)^{-3}v$
   (c) 29 Oct. 2019
   (d) The operator must be L_a instead of L.
5. **p. 632, line 11 from above, the proof of Theorem 5.22:**
   (a) dz, as required.
   (b) dz. Since $(\mathrm{id}_H + L_a)^{-3}w \to (\mathrm{id}_H + L)^{-3}w$ for any $w \in \mathrm{Dom}(L)$ and $\|f(L_a)\|_{B(H)} \leq \|f\|_{L^\infty(\{\mathrm{Re}(z)>0\})}$, we obtain the desired result.
   (c) 29 Oct. 2019
6. **p. 632, Theorem 5.23:**
   (a) $\|f\|_{L^\infty(i\mathbb{R})}$
   (b) $\|f\|_{L^\infty(i\mathbb{R})}$, if there exists $0 < \theta^* < \dfrac{\pi}{2}$ such that $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$.
   (c) 29 Oct. 2019, 15 Sep. 2020
7. **p. 634, line 7 from below:**
   (a) operators L.
   (b) operators L such that there exists $0 < \theta^* < \dfrac{\pi}{2}$ satisfying $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$.

(c) 29 Oct. 2019

8. **p. 635, line 3 from above:**
   (a) $|\sqrt{z+b} - \sqrt{z+a}| =$
   (b) $|\sqrt{z+b} - \sqrt{z+a}| \leq$
   (c) I used the triangle inequality.

9. **p. 635, line 5 from above:**
   (a) we have the following conclusion:
   (b) we have the following conclusion, which defines $\frac{\sqrt{L}}{\mathrm{id}_H + L}$:
   (c) 29 Oct. 2019, 15 Sep. 2020
   (d) On line before Corollary 5.7, I should have clarified that Corollary 5.7 defines $\frac{\sqrt{L}}{\mathrm{id}_H + L}$.

10. **p. 635, Corollary 5.7:**
    (a) L be a sectorial operator.
    (b) L be a sectorial operator, and let $0 < \theta^* < \dfrac{\pi}{2}$ satisfy $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$.
    (c) 29 Oct. 2019, 15 Sep. 2020

11. **p. 635, Definition 5.17:**
    (a) L be a sectorial operator.
    (b) L be a sectorial operator, and let $0 < \theta^* < \dfrac{\pi}{2}$ satisfy $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$.
    (c) 29 Oct. 2019, 15 Sep. 2020

12. **p. 638, Lemma 5.20:**
    (a) L be a sectorial operator.
    (b) L be a sectorial operator, and let $0 < \theta^* < \dfrac{\pi}{2}$ satisfy $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$.
    (c) 29 Oct. 2019, 15 Sep. 2020

13. **p. 638, Lemma 5.21:**
    (a) The operator
    (b) Assume that L is a sectorial operator such that $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$ for some $0 < \theta^* < \dfrac{\pi}{2}$. Then the operator
    (c) 29 Oct. 2019

14. **p. 638, Theorem 5.25:**
    (a) Then there

(b) Assume that L is a sectorial operator such that $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$ for some $0 < \theta^* < \dfrac{\pi}{2}$. Then there

(c) 29 Oct. 2019

15. **p. 639, line 4 from above:**

(a) $\displaystyle\int_0^\infty \frac{a\mathrm{d}x}{1+ax^2} = \sqrt{a}\pi.$

(b) $\displaystyle\int_0^\infty \frac{2a\mathrm{d}x}{1+ax^2} = \sqrt{a}\pi.$

(c) 29 Oct. 2019

(d) $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \pi$ is not true. $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \frac{\pi}{2}$ is correct.

16. **p. 639, Theorem 5.26:**

(a) For all

(b) Assume that L is a sectorial operator such that $|\arg(\langle Lu, u\rangle_H)| < \dfrac{\pi}{2} - \theta^*$ for all $u \in \mathrm{Dom}(L)$ for some $0 < \theta^* < \dfrac{\pi}{2}$. Then for all

(c) 29 Oct. 2019

17. **p. 639, (5.93):**

(a) $\mathrm{id}_H + 2\varepsilon \mathrm{id}_H$

(b) $(1+2\varepsilon)\mathrm{id}_H$

(c) This is just a matter of a marginal typo.

18. p. 639, (5.94)

(a) $\mathrm{id}_H + 2\varepsilon \mathrm{id}_H$

(b) $(1+2\varepsilon)\mathrm{id}_H$

(c) 29 Oct. 2019

(d) This is just a matter of a marginal typo.

1.66 Pages 640-649

1. **p. 640, Corollary 5.8:**

   (a) $\sqrt{L}u = \dfrac{1}{\pi}\displaystyle\int_0^\infty L(\mathrm{id}_H + t^2 L)^{-1} u \mathrm{d}t.$

   (b) $\sqrt{L}u = \dfrac{2}{\pi}\displaystyle\int_0^\infty L(\mathrm{id}_H + t^2 L)^{-1} u \mathrm{d}t.$

   (c) 29 Oct. 2019

   (d) $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \pi$ should have been $\displaystyle\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = \frac{\pi}{2}.$

1.67 Pages 650-659

1. **p. 656, Theorem 5.36:**
    (a) Remove
    (b) Assume that the boundary is smooth.
    (c) 29 Oct. 2019

1.68 Pages 660-669

1. **p. 660, the first line of Lemma 5.29:**
   (a) λ'
   (b) λ
   (c) 29 Oct. 2019
2. **p. 660, (5.120):**
   (a) $\lambda'|\xi|^2$
   (b) $\lambda^2|\xi|^2$
   (c) 29 Oct. 2019
3. **p. 662, lines 3 and 4 from below, the proof of Lemma 5.29:**
   (a) and $3e^{2k} - 8e^k + 4 \geq e^k - 1$ for $k \geq 2$, we conclude
   $$e^\alpha \int_F |u^t(x)|^2 \mathrm{d}x \leq \int_{\mathbb{R}^n} |u^t(x)|^2 \exp(\alpha\eta(x)) \mathrm{d}x \leq \int_{\mathbb{R}^n} |u^t(x)|^2 \mathrm{d}x \leq \int_E |f(x)|^2 \mathrm{d}x.$$
   (b) and
   $$\|u^t \exp(\alpha\eta)\|_{L^2} \leq \|u^t(\exp(\alpha\eta) - 1)\|_{L^2} + \|u^t\|_{L^2} \leq \frac{1}{2}\|u^t \exp(\alpha\eta)\|_{L^2} + \|u^t\|_{L^2},$$
   we conclude
   $$e^\alpha \int_F |u^t(x)|^2 \mathrm{d}x \leq \int_{\mathbb{R}^n} |u^t(x)|^2 \exp(\alpha\eta(x)) \mathrm{d}x \leq 4 \int_{\mathbb{R}^n} |u^t(x)|^2 \mathrm{d}x \leq 4 \int_E |f(x)|^2 \mathrm{d}x.$$
   (c) 23 Nov. 2019

1.69 Pages 670-679

1.70 Pages 680-689

1.71 Pages 690-699

1.72 Pages 700-709

1.73 Pages 710-719

1. **p. 711, (6.1):**
   (a) L^1
   (b) $L^1(\mathbb{R}^n)$
   (c) 14 Sep. 2020
2. **p. 711, Definition 6.2:**
   (a) $f \in X$
   (b) $f \in X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
3. **p. 711, Definition 6.2:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
4. **p. 712, line 3 from above, Definition 6.3:**
   (a) (λ, ∞)
   (b) $(\lambda, \infty]$
   (c) 18 Sep. 2020
   (d) See the remark on the erratum of p. 30.
5. **p. 713:**
   - Exercise 6.4
   - Exercise 6.5 (twice)

   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
6. **p. 716, line 14 from below, the proof of Theorem 6.2:**
   (a) to obtain
   (b) to obtain $\mu\{x \in 10Q : M\mu_1(x)^\delta > \lambda\} \le \min(\mu(10Q), \lambda^{-\delta}|Q|)$.
   (c) 14 Sep. 2024
7.
8. **p. 716, line 13 from below, the proof of Theorem 6.2:**
   (a) to obtain
   (b) to obtain $\mu\{x \in 10Q : M\mu_1(x)^\delta > \lambda\} \le \min(\mu(10Q), \lambda^{-\delta}|Q|)$.
   (c) 14 Sep. 2024

9. **p. 718, Lemma 6.2, the beginning of the proof:**
   (a) We may assume
   (b) Since we work in Q, we may assume
   (c) 14 Sep. 2020
   (d) The weight w here should be considered as a local weight. To the minimum, $\sum_{R \in \mathcal{J}} m_R(w)\chi_R$ is not an A_∞-weight since it vanishes outside Q.
10. **p. 719, line 6 from above (twice), the proof of Lemma 6.2:**
    (a) $m^{(1+\varepsilon)}$
    (b) $m_Q^{(1+\varepsilon)}$
    (c) 15 Sep. 2020
    (d) appears twice

1.74 Pages 720-729

1. **p. 720, line 1 from above:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
2. **p. 720, line 14 from above, Proof of Theorem 6.4:**
   (a) $q = \dfrac{p}{1+\varepsilon}$
   (b) $q = \dfrac{p+\varepsilon}{1+\varepsilon}$
   (c) 29 Oct. 2019
   (d) I should have used $w^{-\frac{1}{p-1}} \in A_{p'} \subset A_\infty$ together with Lemma 6.2.
3. **p. 720, line 16 from above, Proof of Theorem 6.4:**
   (a) there exists $\delta > 0$ such that $\dfrac{w(A)}{w(Q)} \lesssim \left(\dfrac{|A|}{|Q|}\right)^\delta$, whenever A is a subset of a cube Q.
   (b) there exists a constant $D > 0$ such that $\dfrac{w(E)}{w(Q)} \le D\left(\dfrac{|E|}{|Q|}\right)^{\frac{\varepsilon}{1+\varepsilon}}$, whenever E is a subset of a cube Q.
   (c) 29 Oct. 2019
4. **p. 720, line 18 from above, Proof of Theorem 6.4:**
   (a) As a consequence there exists $0 < \alpha_0, \beta_0 < 1$ such that $w(E) \le \beta_0 w(Q)$ for all measurable sets E and Q such that Q is a cube containing E and that $|E| \le \alpha_0|Q|$.
   (b) If we set $\beta_0 = D\alpha_0{}^{\frac{\varepsilon}{1+\varepsilon}}$ with $0 < \alpha_0 \ll 1$, then $w(E) \le \beta_0 w(Q)$ for all measurable sets E and Q such that Q is a cube containing E and that $|E| \le \alpha_0|Q|$.
   (c) 29 Oct. 2019, 16 Sep. 2020
5. **p. 720, line 19, Proof of Theorem 6.4:**
   (a) then there exists $0 < \alpha, \beta < 1$ such that
   (b) then
   (c) 15 Sep. 2020
6. **p. 720, line 19 from above, Proof of Theorem 6.4:**
   (a) If we contrapose this fact, then there exists $0 < \alpha, \beta < 1$ such that
   (b) If we contrapose this fact and set $\alpha \equiv 1 - \beta_0$ and $\beta \equiv 1 - \alpha_0$
   (c) 29 Oct. 2019
7. **p. 720, line 20 from above, Proof of Theorem 6.4:**
   (a) $E \subset Q, \quad w(E) < \alpha' w(Q) \Longrightarrow |E| < \beta'|Q|$
   (b) $E \subset Q, \quad w(E) < \alpha w(Q) \Longrightarrow |E| < \beta|Q|$

(c) 29 Oct. 2019

8. **p. 720, line 25 from above, Proof of Theorem 6.4:**
   (a) Denote by $M_{\rm dyadic,\mathit{Q}}$ the dyadic maximal operator with respect to Q.
   (b) Denote by $M_{\rm dyadic,\mathit{Q},\mathit{w}}$ the dyadic maximal operator with respect to Q with weight w.
   (c) 29 Oct. 2019

9. **p. 720, line 26 from above, Proof of Theorem 6.4:**
   (a) $\{x \in Q \, : \, M_{\rm dyadic,\mathit{Q}} w(x) > \lambda_k\}$
   (b) $\{x \in Q \, : \, M_{\rm dyadic,\mathit{Q},\mathit{w}}[w^{-1}](x) > \lambda_k\}$
   (c) 29 Oct. 2019

10. **p. 721, line 9 from above, Proof of Theorem 6.4:**
    (a) $\le {\lambda_{k+1}}^\tau |\Omega_k|$
    (b) $\le \sum_{k=-1}^{\infty} {\lambda_{k+1}}^\tau |\Omega_k|$
    (c) 29 Oct. 2019

11. **p. 721, line 11 from above, Proof of Theorem 6.4:**
    (a) $\dfrac{1}{w(Q)} \displaystyle\int_Q \dfrac{{\rm d}x}{w(x)^\tau} \lesssim \dfrac{|Q|^\tau}{w(Q)^\tau}$
    (b) $\dfrac{1}{|Q|} \displaystyle\int_Q \dfrac{{\rm d}x}{w(x)^\tau} \lesssim \dfrac{|Q|^\tau}{w(Q)^\tau}$
    (c) 29 Oct. 2019

12. **p. 727, Theorem 6.9:**
    (a) $(1,\infty]^n$
    (b) $(1,\infty)^n$
    (c) 15 Sep. 2020
    (d) Otherwise there is a counterexample. Let me explain why the proof breaks down if $\infty \in \{p_j \, : \, j = 1, 2, \ldots, n\}$. We need to use Theorem 6.8, where we had to assume $p < \infty$. This fact reflects the assumption that $p_n < \infty$. A similar argument must be done for other variables. As a result, we have $p_j < \infty$.

13. **p. 727, line 7 from below:**
    (a) X
    (b) $X(\mathbb{R}^n)$
    (c) 14 Sep. 2020

14. **p. 728, Defintion 6.7:**
    (a) $[1,\infty]$
    (b) $(1,\infty]$

(c) 14 Sep. 2020

15. **p. 728, Defintion 6.7 (three times):**

    (a) ρ_p

    (b) $\rho_{p(\star)}$

    (c) 7 Jul. 2021

1.75 Pages 730-739

1. **p. 732, line 5 from below, the proof of Theorem 6.13:**
   (a) Then $a_j + b_j + c_j = 1$ by the definition.
   (b) Then after a change of equivalent norms we may assume $a_j + b_j + c_j = 1$.
   (c) 15 Sep. 2020
2. **p. 735 (item) 8, one line above (6.57):**
   (a) $p(x) \leq p(y)$
   (b) $p(x) < p(y)$
   (c) 15 Sep. 2020
3. **p. 735, line 1 from below, Theorem 6.15:**
   (a) $\langle x \rangle^{-np_-}$
   (b) $\langle x \rangle^{-n}$
   (c) 15 Sep. 2020
   (d) appears twice
4. **p. 736, line 3 from above, one line below (6.59):**
   (a) We prove.... We note
   (b) We note
   (c) 12 Jan. 2022
5. **p. 737, 5 lines above (6.62):**
   (a) $\chi_{(p(x),\infty)}(p)f$
   (b) $\chi_{(p(x),\infty)}(p(\star))f$
   (c) 15 Sep. 2020
6. **p. 737, 3 lines below (6.62):**
   (a) average
   (b) the average
   (c) 15 Sep. 2020
7. **p. 737, 4 lines below (6.62):**
   (a) $\gamma^2 f$
   (b) $\gamma^4 f$
   (c) 15 Sep. 2020
8. **p. 737, line 7 from below, the proof of Theorem 6.15:**
   (a) (p_-)
   (b) $(\frac{1}{p_-})$

(c) 12 Jan. 2022

9. **p. 738, line 3 from below:**
   (a) $p(\star), q(\star) \to (0,\infty)$
   (b) $p(\star), q(\star) : \mathbb{R}^n \to (0,\infty)$
   (c) 15 Sep. 2020

10. **p. 738, Definition 6.8:**
   (a) $p(\star), q(\star) \to (0,\infty)$
   (b) $p(\star), q(\star) : \mathbb{R}^n \to (0,\infty)$
   (c) 15 Sep. 2020

11. **p. 738, Definition 6.8:**
   (a) Define

$$\|\{f_j\}_{j=1}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \equiv \inf\left(\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n}\sum_{j=1}^{\infty}\frac{|f_j(x)|^{p(x)}}{\lambda^{p(x)/q(x)}}\mathrm{d}x \leq 1\right\} \cup \{\infty\}\right)$$

   (b) Write $r(\cdot) \equiv \frac{p(\cdot)}{q(\cdot)}$. Define

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_j\}_{j=1}^{\infty}) \equiv \sum_{j=1}^{\infty}\inf\left(\left\{\lambda \in (0,\infty) \,:\, \int_{\mathbb{R}^n}\frac{|f_j(x)|^{p(x)}}{\lambda^{r(x)}}\mathrm{d}x \leq 1\right\} \cup \{\infty\}\right)$$

   and

$$\|\{f_j\}_{j=1}^{\infty}\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \equiv \inf\left\{\mu > 0 \,:\, \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{\mu^{-1}f_j\}_{j=1}^{\infty}) \leq 1\right\}$$

   (c) 15 Sep. 2020

12. **p. 738, line 3 from above:**
   (a) (p_-)
   (b) $(\frac{1}{p_-})$
   (c) 12 Jan. 2022

13. **p. 738, Definition 6.8:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020

1.76 Pages 740 - 749

1. **p. 742, line 5 from below, the proof of Lemma 6.13:**
   (a) $m_Q(f_\nu)\chi_{3Q}(x)^{q(x)}$
   (b) $(m_Q(f_\nu)\chi_{3Q}(x))^{q(x)}$
   (c) 12 Jan. 2022
2. **p. 742, line 2 from below, the proof of Lemma 6.13:**
   (a) $m_Q^{(q_-)}(|f_\nu|^{q(\star)})$
   (b) $m_Q^{(\frac{1}{q_-})}({f_\nu}^{q(\star)})$
   (c) 12 Jan. 2022
3. **p. 743, line 3 from above, the proof of Lemma 6.13:**
   (a) $\frac{p(x)q_-}{q(x)}$
   (b) $\frac{p(x)}{q(x)}$
   (c) 12 Jan. 2022
4. **p. 749, line 8 from above, two lines above Theorem 6.20:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020

1.77 Pages 750 - 759

1. **p. 751, line 5 from above, Exercise 6.17:**
   (a) $|x|^{-\frac{n}{p}}$
   (b) $|\star|^{-\frac{n}{p}}$
   (c) 14 Sep. 2020

1.78 Pages 760 - 769

1. **p. 760, one line below (6.100), the proof of Proposition 6.4:**
   (a) the continuous variable
   (b) the continuous variables
   (c) 14 Sep. 2020
2. **p. 761, line 12 from above. one line above Theorem 6.24:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
3. **p. 763, line 5 from above. one line above Definition 6.15:**
   (a) Orlicz
   (b) (Musielak–)Orlicz
   (c) 14 Sep. 2020
4. **p. 768, line 11 from above, the proof of Lemma 6.20:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020

1.79 Pages 770 - 779

1. p. 771, Exercise 6.28

   (a) X

   (b) $X(\mathbb{R}^n)$

   (c) 14 Sep. 2020

2. p. 771, Exercise 6.28

   (a) Y

   (b) $Y(\mathbb{R}^n)$

   (c) 14 Sep. 2020

3. p. 771, Exercise 6.28

   (a) Y^*

   (b) $Y(\mathbb{R}^n)^*$

   (c) 14 Sep. 2020

4. p. 775, two lines above (6.115)

   (a) X

   (b) $X(\mathbb{R}^n)$

   (c) 14 Sep. 2020

5. p. 775, Theorem 6.28

   (a) X^m

   (b) L^m

   (c) 22 Sep. 2022

1.80 Pages 770 - 779

1. **p. 771, Exercise 6.28:**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
2. **p. 771, Exercise 6.28:**
   (a) Y
   (b) $Y(\mathbb{R}^n)$
   (c) 14 Sep. 2020
3. **p. 771, Exercise 6.28:**
   (a) Y^*
   (b) $Y(\mathbb{R}^n)^*$
   (c) 14 Sep. 2020
4. **p. 775, two lines above (6.115):**
   (a) X
   (b) $X(\mathbb{R}^n)$
   (c) 14 Sep. 2020
5. **p. 775, Theorem 6.28:**
   (a) X^m
   (b) L^m
   (c) 22 Sep. 2022

1.81 Pages 780 - 789

1.82 Pages 790 - 799

1.83 Pages 800 - 809

1.84 Pages 810 - 819

1.85 Pages 820 - 829

1.86 Pages 830 - 839

1. **p. 833, (6.187):**

   (a) $\lambda|\xi|^2$

   (b) $\lambda^2|\xi|^2$

   (c) 29 Oct. 2019

2. **p. 837, Exercise 6.90:**

   (a) using (6.191)

   (b) using (6.190) and (6.191)

   (c) 29 Oct. 2019

   (d) I also want to give a hint: Show first that

   $$(\|\nabla f\|_{(L^2)^n})^2 \lesssim |\langle A\nabla f, \nabla f\rangle_{(L^2)^n}|$$

   for all $f \in \mathrm{Dom}(L)$.

1.87 Pages 840 - 849

1. **p. 840, (6.217):**
   (a) $\langle t^{-1}(x-y)\rangle^m$
   (b) $\langle t^{-1}(x-y)\rangle^{2m}$
   (c) 4 Nov. 2019
   (d) This is by no means a mistake but it is somewhat strange.
2. **p. 840, (6.217):**
   (a) $\langle t^{-1}(x-z)\rangle^m$
   (b) $\langle t^{-1}(x-z)\rangle^{2m}$
   (c) 4 Nov. 2019
   (d) This is by no means a mistake but it is somewhat strange.
3. **p. 840, line 12 from above, the proof of Lemma 6.27:**
   (a) $\le t^{-n}$
   (b) $\lesssim t^{-n}$
   (c) 4 Nov. 2019
4. **p. 840, line 12 from above, the proof of Lemma 6.27:**
   (a) $\langle t^{-1}(y-z)\rangle^m$
   (b) $\langle t^{-1}(y-z)\rangle^{2m}$
   (c) 9 Nov. 2019
5. **p. 842, title of § 6.7.2.3:**
   (a) Applications of Gaffney-type estimates to commutators
   (b) Commutator estimates
   (c) 9 Nov. 2019
6. **p. 842, line 2 from below, the proof of Lemma 6.28:**
   (a) $-t\left\{(M_A\nabla h)\cdot t\mathrm{div}(\mathrm{id}_{L^2}+t^2L)^{-1}\right\}$
   (b) $+t\left\{(M_A\nabla h)\cdot t\mathrm{div}(\mathrm{id}_{L^2}+t^2L)^{-1}\right\}$
   (c) 9 Nov. 2019
7. **p. 846, line 3 from below, one line above Lemma 6.33:**
   (a) We first control $U_t\circ P_t$, as follows.
   (b) We define $P_t^n\in B(L^2(\mathbb{R}^n)^n)$ naturally from $P_t\in B(L^2(\mathbb{R}^n))$. We first control $U_t\circ P_t^n$, as follows.
   (c) 9 Nov. 2019
8. **p. 846, Lemma 6.33:**

(a) $U_t \circ P_t$

(b) $U_t \circ P_t^n$

(c) 9 Nov. 2019

9. **p. 847, line 5 from below, the proof of Lemma 6.33:**

(a) $U_t \circ P_t \circ Q_s$

(b) $U_t \circ P_t^n \circ Q_s^n$

(c) 9 Nov. 2019

10. **p. 847, line 4 from below, the proof of Lemma 6.33:**

(a) where Q_s is given by (6.209)

(b) where $Q_s^n \in B(L^2(\mathbb{R}^n)^n, L^2(\mathbb{R}^n)^n)$ is defined naturally from Q_s given by (6.209)

(c) 9 Nov. 2019, 15 Sep. 2020

11. **p. 848, line 3 from above, the proof of Lemma 6.33:**

(a) Let us write E for the identity matrix of size n. We define $P_t^n \in B(L^2(\mathbb{R}^n)^n)$ naturally from $P_t \in B(L^2(\mathbb{R}^n))$, using the identity

(b) Let $g \in \mathrm{Dom}(L)$. We calculate

(c) 9 Nov. 2019

12. **p. 848, lines 4–8 from above, the proof of Lemma 6.33:**

(a)

$$\begin{aligned}
\theta_t \nabla g &= \theta_t \nabla g - \gamma_t \cdot ({P_t}^2 \nabla g) + \gamma_t \cdot ({P_t}^2 \nabla g) \\
&= \theta_t (\mathrm{id}_{(L^2)^n} - (P_t^n)^2) \nabla g + U_t \circ P_t g + \gamma_t \cdot ({P_t}^2 \nabla g) \\
&= \theta_t \nabla (\mathrm{id}_{L^2} - {P_t}^2) g + U_t \circ P_t g + \gamma_t \cdot ({P_t}^2 \nabla g) \\
&= tL(\mathrm{id}_{L^2} + t^2 L)^{-1} (\mathrm{id}_{L^2} - {P_t}^2) g - U_t \circ P_t g + \gamma_t \cdot ({P_t}^2 \nabla g).
\end{aligned}$$

Recall that we considered $U_t \circ P_t$ in Lemma

(b)

$$\begin{aligned}
\theta_t \nabla g &= \theta_t \nabla g - \gamma_t \cdot ({P_t^n}^2 \nabla g) + \gamma_t \cdot ({P_t^n}^2 \nabla g) \\
&= \theta_t (\mathrm{id}_{(L^2)^n} - (P_t^n)^2) \nabla g + U_t \circ P_t^n \nabla g + \gamma_t \cdot ({P_t^n}^2 \nabla g) \\
&= \theta_t \nabla (\mathrm{id}_{L^2} - {P_t^n}^2) g + U_t \circ P_t^n \nabla g + \gamma_t \cdot ({P_t^n}^2 \nabla g) \\
&= tL(\mathrm{id}_{L^2} + t^2 L)^{-1} (\mathrm{id}_{L^2} - {P_t}^2) g - U_t \circ P_t^n \nabla g + \gamma_t \cdot ({P_t}^2 \nabla g).
\end{aligned}$$

Recall that we considered $U_t \circ P_t^n$ in Lemma

(c) 9 Nov. 2019

13. **p. 848, (6.229):**

(a) $\gamma_t \cdot {P_t}^2 \nabla g$

(b) $(\theta_t(\mathbf{e}_1), \theta_t(\mathbf{e}_2), \ldots \theta_t(\mathbf{e}_n)) \cdot (P_t^n)^2 \nabla g$

(c) 9 Nov. 2019

14. **p. 848, line 1 from below, the proof of Proposition 6.10:**

    (a) p. 850 line 4 from above (twice)

    (b) p. 850 line 5 from above

    (c) ${P_t}^2\nabla g$

    (d) $(P_t^n)^2\nabla g$

    (e) 9 Nov. 2019

15. (a) p. 849, (6.231), the proof of Lemma 6.34

    (b) p. 849, lines 4 and 7 from below, the proof of Lemma 6.34

    (a) $\|\theta_t\mathbf{G} - \gamma_t \cdot S_t^n\mathbf{G}\|_2$

    (b) $\|\theta_t\mathbf{G} - (\theta_t(\mathbf{e}_1), \theta_t(\mathbf{e}_2), \dots, \theta_t(\mathbf{e}_n)) \cdot S_t^n\mathbf{G}\|_2$

    (c) 9 Nov. 2019

16. p. 849, lines 4 and 7 from below, the proof of Lemma 6.34

    (a) $\gamma_t \cdot S_t^n\mathbf{G}$

    (b) $(\theta_t(\mathbf{e}_1), \theta_t(\mathbf{e}_2), \dots, \theta_t(\mathbf{e}_n)) \cdot S_t^n\mathbf{G}$

    (c) 9 Nov. 2019

17. p. 849, line 11 from below, the proof of Lemma 6.34

    (a) $\gamma_t \cdot S_t^n$

    (b) $(\theta_t(\mathbf{e}_1), \theta_t(\mathbf{e}_2), \dots, \theta_t(\mathbf{e}_n)) \cdot S_t^n$

    (c) 9 Nov. 2019

18. p. 849, line 7 from below, the proof of Lemma 6.34

    (a) $\gamma_t(x) \cdot m_Q(\mathbf{G})$

    (b) $(\theta_t(\mathbf{e}_1)(x), \theta_t(\mathbf{e}_2)(x), \dots, \theta_t(\mathbf{e}_n)(x)) \cdot S_t^n\mathbf{G}$

    (c) 9 Nov. 2019

19. p. 849, (6.231), lines 4 and 7 from below, the proof of Lemma 6.35

    (a) $\|\theta_t\mathbf{G} - \gamma_t \cdot S_t^n\mathbf{G}\|_2$

    (b) $\|\theta_t\mathbf{G} - (\theta_t(\mathbf{e}_n), \theta_t(\mathbf{e}_1), \dots, \theta_t(\mathbf{e}_n)) \cdot S_t^n\mathbf{G}\|_2$

    (c) 9 Nov. 2019

20. p. 849, line 7 from below, the proof of Lemma 6.35

    (a) $\gamma_t(x) \cdot m_Q(\mathbf{G})$

    (b) $(\theta_t(\mathbf{e}_1)(x), \theta_t(\mathbf{e}_2)(x), \dots, \theta_t(\mathbf{e}_n))(x) \cdot m_Q(\mathbf{G})$

    (c) 9 Nov. 2019

1.88 Pages 850 - 859

1. **p. 850, lines 5, 8 and 11 from above, the proof of Lemma 6.35:**
   (a) $\gamma_t(x) \cdot S_t \nabla f^{\varepsilon}_{Q,w}(x)$
   (b) $(\theta_t(\mathbf{e}_1)(x), \theta_t(\mathbf{e}_2)(x), \ldots, \theta_t(\mathbf{e}_n))(x) \cdot S_t \nabla f^{\varepsilon}_{Q,w}(x)$
   (c) 9 Nov. 2019
2. **p. 851, line 13 from above, the proof of Lemma 6.35:**
   (a) $\|_2$
   (b) $\|_{(L^2)^n}$
   (c) 9 Nov. 2019
3. **p. 895:**
   (a) I listed
      i. Sagher, Y., Zhou, K.C.: A local version of a theorem of Khinchin. In: Analysis and Partial Differential Equations. Lecture Notes in Pure and Applied Mathematics, vol. 122, pp. 327–330. Dekker, New York (1990).
   (b) But this should have been appeared elsewhere.
   (c) 29 Oct. 2019

2 Missing lemmas/assertions

Here I will collect what I should have added in my book.

2.1 Remark on Theorem 2.4

We will show that

$$\psi_k \cdot g \to f$$

as $k \to \infty$ in $B^s_{pq}(\mathbb{R}^n)$.

1. Choose $\psi, \varphi \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ so that

$$\chi_{B(4)} \le \psi \le \chi_{B(8)}, \chi_{B(4)\setminus B(2)} \le \varphi \le \chi_{B(8)\setminus B(1)}.$$

Let $\tau \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $f \in B^s_{pq}(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $0 < p, q \le \infty$. Then $\tau(D)f \in B^S_{pq}(\mathbb{R}^n)$ for all $S \in \mathbb{R}$.

Indeed, using

$$\|f\|_{B^s_{pq}} = \|\psi(D)f\|_{L^p} + \left(\sum_{j=1}^{\infty}(2^{js}\|\varphi_j(D)f\|_{L^p})^q\right)^{\frac{1}{q}} < \infty,$$

we need to show that

$$\|\tau(D)f\|_{B^S_{pq}} = \|\tau(D)\psi(D)f\|_{L^p} + \left(\sum_{j=1}^{\infty}(2^{jS}\|\tau(D)\varphi_j(D)f\|_{L^p})^q\right)^{\frac{1}{q}} < \infty.$$

Since τ is compactly supported, there exists an integer $K \in \mathbb{N}$ such that $\mathrm{supp}(\tau) \subset B(2^K)$. Note that $\mathrm{supp}(\varphi_j) \subset B(2^{j+3}) \setminus B(2^j)$. Thus, if $j > K$, then $\tau\varphi_j = 0$, implying that $\tau(D)\varphi_j(D)f = 0$ for such j. Thus,

$$\|\tau(D)f\|_{B^S_{pq}} = \|\tau(D)\psi(D)f\|_{L^p} + \left(\sum_{j=1}^{K}(2^{jS}\|\tau(D)\varphi_j(D)f\|_{L^p})^q\right)^{\frac{1}{q}}.$$

Let

$$M = \max_{j=0,1,2,\ldots,K} 2^{j(S-s)} \in [1,\infty).$$

Then

$$\|\tau(D)f\|_{B^S_{pq}} \le M\|\tau(D)\psi(D)f\|_{L^p} + M\left(\sum_{j=1}^{K}(2^{js}\|\tau(D)\varphi_j(D)f\|_{L^p})^q\right)^{\frac{1}{q}}. \tag{2.1}$$

Finally by the Young inequality, we have

$$\|\tau(D)\psi(D)f\|_{L^p} \simeq \|\mathcal{F}^{-1}\tau * \psi(D)f\|_{L^p} \le \|\mathcal{F}^{-1}\tau\|_{L^1}\|\psi(D)f\|_{L^p} \simeq \|\psi(D)f\|_{L^p}$$

and

$$\|\tau(D)\varphi_j(D)f\|_{L^p} \simeq \|\mathcal{F}^{-1}\tau * \varphi_j(D)f\|_{L^p} \le \|\mathcal{F}^{-1}\tau\|_{L^1}\|\varphi_j(D)f\|_{L^p} \simeq \|\varphi_j(D)f\|_{L^p}.$$

Inserting these inequalities into (2.1), we obtain

$$\|\tau(D)f\|_{B^S_{pq}} \lesssim M\|\psi(D)f\|_{L^p} + M\left(\sum_{j=1}^{K}(2^{js}\|\varphi_j(D)f\|_{L^p})^q\right)^{\frac{1}{q}} \le M\|f\|_{B^s_{pq}} < \infty.$$

This proves the claim.

2. Note that $B^{s+2}_{pq}(\mathbb{R}^n)$ is contained in $W^{[s+1]}_p(\mathbb{R}^n)$ according to Proposition 2.1 and Theorem 2.3. It follows that g belongs to $W^{[s+1]}_p(\mathbb{R}^n)$.
3. It is straightforward to check that $g - \psi_k g$ converges to 0 in $W^{[s+1]}_p(\mathbb{R}^n)$ using that g belongs to $W^{[s+1]}_p(\mathbb{R}^n)$.
4. Since $W^{[s+1]}_p(\mathbb{R}^n)$ is contained in $B^{[s+1]-\delta}_{p1}(\mathbb{R}^n)$ according to Propositions 2.1 and 2.3 and Theorem 2.3, $g - \psi_k g$ converges to 0 in $B^{[s+1]-\delta}_{p1}(\mathbb{R}^n)$. Here $\delta > 0$ is sufficiently small.
5. If we use Proposition 2.3 once again, we see that the convergence takes place in $B^s_{pq}(\mathbb{R}^n)$.

2.2 p. 628

In addition to the conclusions presented in Theorem 5.21, I should have proved that

$$|\arg(\langle D^*ADx, x\rangle_H)| < \frac{\pi}{2} - \theta \quad (x \in \mathrm{Dom}(D^*AD))$$

for some $0 < \theta < \dfrac{\pi}{2}$ independent of $x \in \mathrm{Dom}(D^*AD)$. This is achieved as follows:

Let $x \in \mathrm{Dom}(D^*AD)$. Then

$$\langle D^*ADx, x\rangle_H = \langle ADx, Dx\rangle_H,$$

since $ADx \in \mathrm{Dom}(D^*)$. Since

$$\Re(\langle ADx, Dx\rangle_H) \gtrsim (\|Dx\|_H)^2, \quad |\langle ADx, Dx\rangle_H| \sim (\|Dx\|_H)^2,$$

we obtain the desired result.

29 Oct. 2019

2.3 p. 632

(5.84) should have been proved. I omitted the proof. However, this can be found in the proof of Theorem 5.23.

29 Oct. 2019

2.4 p. 834

I would like to explain why (6.190) and (6.191) yield (6.192). It will be a solution to Exercise 6.90 at the same time.

This can be explained as follows: Since $H^1(\mathbb{R}^n) \subset \mathrm{Dom}(\sqrt{L}) \cap \mathrm{Dom}(\sqrt{L^*})$, we have

$$(\|\nabla f\|_{(L^2)^n})^2 \lesssim |\langle A\nabla f, \nabla f\rangle_{(L^2)^n}|$$

for all $f \in \mathrm{Dom}(L)$. If $f \in \mathrm{Dom}(L)$, then we have

$$(\|\nabla f\|_{(L^2)^n})^2 \lesssim |\langle Lf, f\rangle_{(L^2)^n}| = |\langle \sqrt{L}\sqrt{L}f, f\rangle_{(L^2)^n}| = |\langle \sqrt{L}f, \sqrt{L^*}f\rangle_{(L^2)^n}|,$$

where we have used $\sqrt{L^*} = (\sqrt{L})^*$ for the last inequality. By virtue of (6.191), we obtain

$$(\|\nabla f\|_{(L^2)^n})^2 \lesssim \|\sqrt{L}f\|_{L^2}\|\nabla f\|_{(L^2)^n}$$

for all $f \in \mathrm{Dom}(L)$. Thus,

$$\|\sqrt{L}f\|_{L^2} \lesssim \|\nabla f\|_{(L^2)^n}$$

for all $f \in \mathrm{Dom}(L)$. If we are given $f \in \mathrm{Dom}(\sqrt{L})$, then we consider $f_j \equiv (\mathrm{id}_H + j^{-1}L)^{-1}f$ for $j \in \mathbb{N}$. We know that

$$\|\nabla f_j - \nabla f_k\|_{(L^2)^n} \sim \|\sqrt{L}(f_j - f_k)\|_{L^2}.$$

Thanks to Lemma 5.21, $\sqrt{L}f_j \to \sqrt{L}f$. Thus, $\{\nabla f_j\}_{j=1}^{\infty}$ converges in $(L^2(\mathbb{R}^n))^n$. Since $f_j \to f$ in $L^2(\mathbb{R}^n)$ as is seen from

$$f_j - f = \sqrt{L}(\mathrm{id}_H + j^{-1}L)^{-1}\sqrt{L}f = \mathrm{O}(j^{-\frac{1}{2}}),$$

we obtain $f \in \mathrm{Dom}(\nabla) = H^1(\mathbb{R}^n)$ and $\nabla f_j \to \nabla f$. Thus by taking the limit in

$$\|\nabla f_j\|_{(L^2)^n} \sim \|\sqrt{L}f_j\|_{L^2},$$

we obtain $\|\nabla f\|_{(L^2)^n} \sim \|\sqrt{L}f\|_{L^2}$.

29 Oct. 2019

3 References

1. See [1] for the Hesteness extension.
2.
3.

References

[1] Hestenes, M. R., *Extension of the range of a differentiable function*. Duke Math. J. **8**, (1941). 183-192. <109>