A translation of the preface of my book

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I aim in this book to explain Besov spaces and Triebel-Lizorkin spaces from the start and to apply them to PDEs.

I mean here by a function space a linear subspace of the space of all functions in a set X. We often place ourselves in the setting of $\mathbb{R}^n$ but in the last part of the book we consider open sets in $\mathbb{R}^n$. For example, the set of all functions on $X = \mathbb{R}^n$ is an example of function spaces. However, it is too vague and too hard to grasp. As more typical examples, let us consider the subspace of all Borel measurable functions in particular the space of all continuous functions. Or, the readers may envisage the space $\mathrm{BC}(\mathbb{R}^n)$ the set of all bounded continuous functions which the readers took up in the lecture of topological spaces. If the readers are familiar with theory of integral, we can think that we are going to study a new framework containing the set of all integrable or measurable fucntions.

There are too many continuous or measurable functions. Therefore, there are various linear subspace of them. In this book, we are going to propose new frameworks called Besov spaces and Triebel-Lizorkin spaces and then discuss in details the property of them.

When we were high school students or undergraduate students, our main concern lay in the C^∞-functions. Therefore, the readers can not understand why we are going to investigate function in a very subtle manner. However, there do exist many examples where the non-differentiable functions play a fundamental role in the rule of the nature.

As an example, I take up the Brownian motion. This is familiar because we learnt that this describes the motion of particles in the chemical course. Here by the Brownian motion, I mean the "mathematical" one which grew out of the chemical one. This mathematical Brownian motion is a fundamental concept of stochastic integral by Kiyosi Ito and hence we are convinced that the mathematical Brownian motion plays a fundamental role in economy. As well as the chemical one, the mathematical one moves in a very complicated way. Namely, each path is continuous but it is too complicated and non-differentiable. When we want to describe properties of continuous functions which are not differentiable, Besov spaces and Triebel-Lizorkin spaces are useful. As this example shows, Besov spaces and Triebel-Lizorkin spaces play an important role.

Another aim of this book is to apply Besov spaces and Triebel-Lizorkin spaces to PDEs.

In PDE, we are led to consider the equations beyond the framework of C^k

functions. Namely, it does not suffice to consider functions which are k-times differentiable and whose partial derivatives up to order k are all continuous. For example, when we consider the elliptic differential equations, we use not C^2 but $\mathcal{C}^{2+\varepsilon}$ with $\varepsilon \in (0,1)$, where $\mathcal{C}^{2+\varepsilon}$ denotes the Hölder-Zygmund space of order $2+\varepsilon$. As we establish later in this book, the space $\mathcal{C}^{2+\varepsilon}$ is realized as a spacial case of Besov spaces and Triebel-Lizorkin spaces. Hence we see that Besov spaces and Triebel-Lizorkin spaces are useful in PDEs. PDE being too wide, we cannot take up all of them but we seek to investigate the wave equations, the Schr"odinger equations, the heat equations and the elliptic differential equations in the context of applications of Besov spaces and Triebel-Lizorkin spaces.

For PDEs, we must be familiar with functional analysis. However, we need not be so serious: we provide some facts on functional analysis in this book and this is sufficient.

Many mathematicians dislike Besov spaces and Triebel-Lizorkin spaces. Why ? I now consider the reason here. As I mentioned earlier, there are many function spaces. Although I content myself with listing them briefly, there are L^p-spaces, Sobolev spaces, Morrey spaces, Orlicz spaces, Besov spaces, Triebel-Lizorkin spaces and so on. Among them I am led to the following conclusion: The more easier to define the function spaces are, the fewer good properties they have. The spaces L^p-spaces, Sobolev spaces, Morrey spaces, Orlicz spaces, which I listed earlier, are easy to describe but they cannot cover differentiablity and integrability. Meanwhile, Besov spaces and Triebel-Lizorkin spaces describes very well differentiablity and integrability but their definitions are very complicated. For example, we can consider Besov-Morrey spaces which are defined by mixing Besov spaces and Morrey spaces. You can easily guess that the definition of Besov-Morrey spaces is very complicated. Many people feel that Besov spaces and Triebel-Lizorkin spaces because of the complexity of the definition which arises as a price of the good properties, I think.

However, let me stress that these function spaces have many big advantages. As I mentioned before, it is important to be able to grasp many other function spaces. For example, L^p-spaces, the BMO space, Sobolev spaces and Hardy spaces fall under the unified framework of Besov spaces and Triebel-Lizorkin spaces. Here I content myself with mentioning that the atomic decomposition is extremely important. Because we can learn much more once we establish the theory of the atomic decomposition. The details are left to Chapter 5.

Let me describe the structure of this book. We have to prepare a lot in order to define Besov spaces and Triebel-Lizorkin spaces and in order to establish the theory because Besov spaces and Triebel-Lizorkin spaces are very complicated. However, if we are limited to B^s_{pq} with $1 \le p,q \le \infty$ and $s \in \mathbb{R}$, we can define the space and investigate the property without using any heavy tool except $\mathcal{S}$ and $\mathcal{S}'$. So in Chapter 1, for the purpose of a survey of the book, I investigate B^s_{pq} with $1 \le p,q \le \infty$ and $s \in \mathbb{R}$. Here I do not go into the detail of Fourier analysis and theory of integration. Instead, I indicate where to look for these preliminary facts in this book. In Chapter 2, I explain about the Fourier transform, the maximal operator and the singular integral operators, which are of fundamental importance in harmonic analysis. The theory of singular integral operators being too wide with a large amount of literature, I content myself with its brief introduction. Chapter 3 is one of my aims in this book. With the culmination of Chapter 2, we define Besov spaces and Triebel-Lizorkin spaces and then we investigate fundamental properties; density of the spaces, completeness of the spaces and so on. Chapter 4 is devoted to showing that these spaces cover many other spaces as special cases. Atomic decomposition is taken up in Chapter 5 together with applications to the boundedness of operators. Applications to PDEs are contained in Chapter 6.

The key theorems in this book are Theorem 2.2.15, its corollary Theorem 2.2.38, Theorem 3.3.6 and its corollary Theorem 5.1.6. Needless to say, the definition is the most important in mathematics. However, these four theorems appear repeatedly and we cannot follow the proof without them. By this I do not mean that we have to memorize these theorems but we will use them quite often.

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My earlier manuscript was not so polished and because of this my presentation of the seminar was poor. I apologize here for making the participant of my seminar puzzled and thank them for that.