REFINEMENT OF STRICHARTZ ESTIMATES FOR AIRY EQUATION AND APPLICATION

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1. Introduction

This is a joint work with Satoshi Masaki (Osaka university). We consider the space-time estimates for the solution $e^{-t\partial_x^3}f$ to the Airy equation

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & t, x \in \mathbb{R}, \\ u(0,x) = f(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function and $f : \mathbb{R} \to \mathbb{R}$ is a given data. As with the Schrödinger equation, the Strichartz estimate for (1.1) is well-known (see [2] for instance). The refinement of the Strichartz estimate for (1.1) is studied by several authors. One of the refinement of the Strichartz estimate is the following Stein-Tomas type estimate.

Theorem 1.1 (Stein-Tomas type estimate [1])**.** *Let $4/3 < p \leqslant \infty$. Then, there exists a positive constant C depending only on p such that the inequality*

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L^{3p}_{t,x}} \leqslant C\|f\|_{\hat{L}^p} \tag{1.2}$$

holds for any $f \in \hat{L}^p$, where the space $\hat{L}^p$ is defined for $1 \leqslant p \leqslant \infty$ by

$$\hat{L}^p = \hat{L}^p(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) | \, \|f\|_{\hat{L}^p} = \|\hat{f}\|_{L^{p'}} < \infty\},$$

where $\hat{f}$ stands for Fourier transform of f with respect to space variable and p' denotes the Hölder conjugate of p.

We consider an improvement of the Stein-Tomas type estimate (1.2). We now introduce generalized Morrey space and generalized hat-Morrey space.

Definition 1.2. *Let $\mathcal{D} := \{\tau_k^j = [k2^{-j}, (k+1)2^{-j}) \mid j, k \in \mathbb{Z}\}$.*
(i) For $1 \leqslant q \leqslant p \leqslant \infty$ and for $1 \leqslant r \leqslant \infty$, we define a generalized Morrey norm $\|\cdot\|_{M^p_{q,r}}$ *by*

$$\|f\|_{M^p_{q,r}} = \left\| |\tau_k^j|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(\tau_k^j)} \right\|_{\ell^r_{j,k}},$$

where, the case $p = q$ and $r < \infty$ is excluded.
(ii) For $1 \leqslant p \leqslant q \leqslant \infty$ and for $1 \leqslant r \leqslant \infty$, we also introduce $\|f\|_{\hat{M}^p_{q,r}} := \|\hat{f}\|_{M^{p'}_{q',r}}$, i.e.,

$$\|f\|_{\hat{M}^p_{q,r}} = \left\| |\tau_k^j|^{\frac{1}{q}-\frac{1}{p}} \left\|\hat{f}\right\|_{L^{q'}(\tau_k^j)} \right\|_{\ell^r_{j,k}}.$$

Banach spaces $M^p_{q,r}$ and $\hat{M}^p_{q,r}$ are defined as sets of tempered distributions of which above norms are finite, respectively.

The first main theorem is as follows.

Theorem 1.3 (Refinement of Stein-Tomas type estimate -diagonal case-[5])**.** *Let $4/3 \leqslant p < \infty$. Then, there exists a positive constant C depending only on p such that the inequality*

$$\left\| |\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f \right\|_{L^{3p}_{t,x}} \leqslant C \, \|f\|_{\hat{M}^p_{\frac{3p}{2},2(\frac{3p}{2})'}} \tag{1.3}$$

holds for any $f \in \hat{M}^p_{\frac{3p}{2},2(\frac{3p}{2})'}$.

Using the refined Stein-Tomas estimate (1.3), we are able to prove the existence of a minimal non-scattering solution to the generalized Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{2\alpha} u) & t, x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{gKdV}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given data, and $\mu = \pm 1$ and $\alpha > 0$ are constants, see [5] for detail.

Furthermore, we obtain the refinement of Stein-Tomas type estimate for the non-diagonal case.

Theorem 1.4 (Refinement of Stein-Tomas type estimate -non-diagonal case-)**.** *Let $\varepsilon \in (0, 1/4)$. Let (p, q) satisfy*

$$0 \leqslant \frac{1}{p} \leqslant \frac{1}{4} - \varepsilon, \qquad \frac{1}{q} \leqslant \frac{1}{2} - \frac{1}{p} - \varepsilon.$$

Define β and s by

$$\frac{2}{p} + \frac{1}{q} = \frac{1}{\beta} - \varepsilon, \qquad s = -\frac{1}{p} + \frac{2}{q} - \varepsilon.$$

Further, we define γ and δ by

$$\frac{1}{\gamma} = \frac{1}{\beta} - \frac{1}{\max(p,q)}, \qquad \frac{1}{\delta} = \frac{1}{2} - \frac{1}{\max(p,q)},$$

then, there exists a positive constant C depending on p, q such that for any $f \in \hat{M}^\beta_{\gamma,\delta}$,

$$\left\| |\partial_x|^s e^{-t\partial_x^3} f \right\|_{L^p_x(\mathbb{R}; L^q_t(\mathbb{R}))} \leqslant C \|f\|_{\hat{M}^\beta_{\gamma,\delta}}.$$

References

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