Well-posedness of some reduced nonlinear dispersive equations in Sobolev spaces with negative indices

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We study the time local well-posedness in low regularity of the Cauchy problem for the nonlinear evolution equation with third order and second order dispersions:

$$\partial_t u + \alpha\partial_x^3 u + i\beta\partial_x^2 u + i\gamma_1|u|^2 u + \gamma_2|u|^2\partial_x u = 0, \tag{1}$$
$$t \in [-T, T], \quad x \in \mathbf{T},$$
$$u(0,x) = u_0(x), \qquad x \in \mathbf{T}, \tag{2}$$

where α, β, γ_1 and γ_2 are real constants and T is a positive constant. Throughout this note, we assume that

$$\alpha\beta \neq 0 \implies 2\beta/3\alpha \notin \mathbf{Z}. \tag{3}$$

We consider the following three cases:

$$\begin{array}{ll}
\text{(cNLS)} & \alpha = 0,\ \beta \neq 0,\ \gamma_1 \neq 0,\ \gamma_2 = 0,\\
\text{(mKdV)} & \alpha \neq 0,\ \beta = 0,\ \gamma_1 = 0, \gamma_2 \neq 0,\\
\text{(3rdLL)} & \alpha \neq 0,\ \beta \neq 0,\ \gamma_1 \neq 0,\ \gamma_2 = 0.
\end{array}$$

In the cases (cNLS), (mKdV) and (3rdLL), equation (1) is called the cubic nonlinear Schrödinger equation, the (complex) modified KdV equation and the third order Lugiato-Lefever equation, respectively. In the physical context, equation (1) often includes the damping term, the detuning term $i\theta u$ ($\theta \in \mathbf{R}$), the external forcing term and others (see [2], [14] and [20]), but we omit those terms because it does not matter as far as the time local well-posedness is concerned. From a scaling point of view, the critical Sobolev spaces are $\dot{H}^{-1/2}$, $\dot{H}^{-1/2}$ and $\dot{H}^{-1}$, respectively, in the cases (cNLS), (mKdV) and (3rdLL). These three equations have two features in common, that is, the cubic nonlinearity and the conservation of L^2 norm. These yield common resonant frequencies though they are different in linear dispersion and nonlinearity (as a result, different in the critical Sobolev spaces) from each other. To be more specific, the so-called reduced equations corresponding to these three cases (cNLS), (mKdV) and (3rdLL) are similar in resonance structure to each other. If we put

$$v(t,x) = u\big(t, x + \frac{\gamma_2}{\pi}\int_0^t \|u(s)\|_{L^2}^2\, ds\big) e^{\frac{\gamma_1}{\pi} i \int_0^t \|u(s)\|_{L^2}^2\, ds + \frac{\gamma_2}{2\pi} i \int_0^t \mathrm{Im}\,(\partial_x u(s), u(s))_{L^2}\, ds},$$

then the reduced equation can be formally written as follows.

$$\begin{gathered} \partial_t v + \alpha \partial_x^3 v + i\beta \partial_x^2 v + i\gamma_1 \big(|v|^2 - \tfrac{1}{\pi}\|v(t)\|_{L^2}^2\big) v \\ + \gamma_2 \big[\big(|v|^2 - \tfrac{1}{2\pi}\|v(t)\|_{L^2}^2\big)\partial_x v - \tfrac{i}{2\pi} \mathrm{Im}\,(\partial_x v, v)_{L^2} v\big] = 0, \\ t \in [-T, T], \quad x \in \mathbf{T}, \end{gathered} \tag{4}$$

where $(\cdot,\cdot)_{L^2}$ denotes the scalar product in $L^2(\mathbf{T})$.

Remark 0.1. We note that the quantities $\|u(s)\|_{L^2}^2$ and $\mathrm{Im}\,(\partial_x u, u)_{L^2}$ correspond to the mass and the current (or the momentum), respectively, for equation (1), which are independent of time t.

In this talk, I first give a brief survey of known results on (cNLS) and (mKdV) and next present several new results on (3rdLL), which are partly based on the joint work with Miyaji Tomoyuki, Meiji Institute for Advanced Study of Mathematical Sciences, Meiji University.

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