リプシッツ作用素の半群

ー導入・生成・近似ー

Semigroups of Lipschitz Operators

— Introduction, Generation and Approximation —

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1.Introduction. Let X be a real Banach space with norm $|\cdot|$ and D a subset of X.

Definition 1.1. A one-parameter family $\{T(t)\}_{t\in[0,\infty)}$ of operators from D into itself is called a **semigroup (of operators)** on D if it satisfies the following conditions (S1) and (S2).

(S1) For $x \in D$ and $t, s \geq 0$,

$$T(t)T(s)x = T(t+s)x, \qquad T(0)x = x.$$

(S2) For $x \in D$ and $t \in [0,\infty)$,

$$\lim_{s\to t}|T(t)x - T(s)x| = 0.$$

It is called a **semigroup of Lipschitz operators** on D if, in addition to (S1) and (S2), it satisfies the following condition (S3).

(S3) For $\tau > 0$, there exists $M_\tau \geq 1$ such that

$$|T(t)x - T(t)y| \leq M_\tau|x - y|$$

for x, $y \in D$ and $t \in [0,\tau]$.

Definition 1.2. Let $\{T(t)\}_{t\in[0,\infty)}$ be a semigroup of operators on D. The operator A_0 defined by $D(A_0) = \left\{x \in D;\ \lim_{h\downarrow 0} h^{-1}(T(h)x - x) \text{ exists}\right\}$ and $A_0 x = \lim_{h\downarrow 0} h^{-1}(T(h)x - x)$ is called the **infinitesimal generator** of the semigroup.

Theorem 1.1. *Let $\{T(t)\}_{t\in[0,\infty)}$ be a semigroup on D. It is a semigroup of Lipschitz operators on D if and only if there exist $M \geq 1$ and $\omega \in \boldsymbol{R}$ such that*

$$|T(t)x - T(t)y| \leq Me^{\omega t}|x-y| \tag{1.1}$$

for $t \in [0,\infty)$ and $x, y \in D$.

Proof. Let $|T(t)|_{Lip}$ denote the least Lipschitz constant of $T(t)$ and

$$f(t) = \begin{cases} \log |T(t)|_{Lip}, & |T(t)|_{Lip} \neq 0 \\ -\infty, & |T(t)|_{Lip} = 0 \end{cases}$$

for $t \in [0,\infty)$. Then, we see that $f(\cdot)$ is Locally bounded and subadditive on $[0,\infty)$, which implies the necessity. Q.E.D.

Remark 1.1. If, for some $\omega \in \boldsymbol{R}$,

$$|T(t)x - T(t)y| \leq e^{\omega t}|x-y|, \ t \in [0,\infty), \ x, y \in D,$$

then the semigroup $\{T(t)\}_{t\in[0,\infty)}$ is called to be *quasi-contractive* with respect to the norm $|\cdot|$ on D.

Theorem 1.2. *Let $\{T(t)\}_{t\in[0,\infty)}$ be a semigroup on D. It is a semigroup of Lipschitz operators on D if and only if there exist $M \geq 1$, $\omega \in \boldsymbol{R}$ and a metric d on D such that*

$$|x-y| \leq d(x,y) \leq M|x-y|, \tag{1.2}$$
$$d(T(t)x, T(t)y) \leq e^{\omega t} d(x,y), \tag{1.3}$$

for $x,\ y \in D$.

Proof. To see the necessity, choose $M \geq 1$ and $\omega \in \boldsymbol{R}$ so that (1.1) holds, and then set

$$d(x,y) = \sup\left\{e^{-\omega t}|T(t)x - T(t)y|;\ t \in [0,\infty)\right\}$$

for $x, y \in D$. Q.E.D.

Remark 1.2. There exists a nonnegative functional $V(\cdot,\cdot)$ on $X \times X$ such that

$$|V(x,y) - V(\hat{x},\hat{y})| \leq M\left(|x-y| + |\hat{x}-\hat{y}|\right),$$
$$(x,y),\ (\hat{x},\hat{y}) \in X \times X,$$
$$V(x,y) = d(x,y),\ (x,y) \in D \times D.$$

Remark 1.3. Let $\hat{D}$ denote the set of $x \in D$ such that $t \to T(t)x$ is locally Lipschitz continuous on $[0,\infty)$. For $x \in \hat{D}$, there exists $L(x) = \lim_{h\downarrow 0} d(T(h)x, x)/h$ and

$$d(T(t)x, T(s)x) \leq L(x)\left|\int_s^t e^{\omega\sigma}\,d\sigma\right|,\ t,s \geq 0.$$

Furthermore, if $x \in \hat{D}$ and $t \geq 0$, then $T(t)x \in \hat{D}$ and $L(T(t)x) \leq e^{\omega t}L(x)$.

Example 1.1. Let f be a continuous function $\boldsymbol{R}$ such that $1/M \leq f(r) \leq 1$ for $r \in \boldsymbol{R}$, where $M \geq 1$ is a constant. Define $T(t)x = u(t)$, where $u(t)$ is the unique C^1 solution u of the initial value problem

$$\partial_t u = f(u), \ u\Big|_{t=0} = x \in \boldsymbol{R},$$

which is given by

$$u(t) = g^{-1}(t + g(x)), \ g(r) = \int_0^r \frac{dr}{f(r)}.$$

The semigroup $\{T(t)\}_{t\in[0,\infty)}$ on $\boldsymbol{R}$ satisfies (1.2), (1.3) with $\omega = 0$ and the metric $d(\cdot,\cdot)$ on $\boldsymbol{R}$ defined by $d(x,y) = |g(x) - g(y)|$ for $x, y \in \boldsymbol{R}$. But, for example, for $f(r) = (1/M + \sqrt{|r|}) \wedge 1$, there is no number $\omega \in \boldsymbol{R}$ such that $\text{sign}(x-y)(f(x) - f(y)) \leq \omega|x-y|$ for $x, y \in \boldsymbol{R}$ and $|T(t)x - T(t)y| \leq e^{\omega t}|x-y|$ for $x, \ y \in \boldsymbol{R}$. Indeed, $f(r) = 1/M + \sqrt{|r|}$ for $|r| \leq \sqrt{1-1/M}$ and $\text{sign}(x-y)(f(x) - f(y)) = |\sqrt{|x|} - \sqrt{|y|}|$ for $|x|, |y| \leq \sqrt{1-1/M}$.

Consider the evolution equation

$$(1.4) \qquad \partial_t u = Au, \quad t > 0$$

in X. Assume that, for $x \in D(A)$, there exists a norm $\|\cdot\|_x$ on X such that

$$|\xi| \le \|\xi\|_x \le M|\xi|, \quad x \in D(A), \quad \xi \in X$$

and $(x, \xi) \to \|\xi\|_x$ is continuous on $D(A) \times X$. Assume also, for any $x, y \in D(A)$, there exists a smooth curve $c = c(\theta)$, $0 \le \theta \le 1$, in $D(A)$ such that $c(0) = x$ and $c(1) = y$. Denote by $\Gamma(x, y)$ the set of such a curve $c = c(\theta)$, $0 \le \theta \le 1$, and define

$$d(x, y) = \inf \left\{ \int_0^1 \|c'(\theta)\|_{c(\theta)} \, d\theta; \ c \in \Gamma(x, y) \right\}.$$

Then, $d(\cdot, \cdot)$ is a metric on $D(A)$ such that $|x - y| \le d(x, y)$ for $x, y \in D(A)$. It holds that $d(x, y) \le M|x - y|$ for $x, y \in D(A)$ if $D(A)$ is convex.

Let $c \in \Gamma(x, y)$ and $u(t, \theta)$ the solution of (1.4) such that $u(0, \theta) = c(\theta)$, $0 \le \theta \le 1$. Assume $u(t, \theta)$ is smooth and set $\xi(t, \theta) = \partial_\theta u(t, \theta)$. Differentiating the both sides of (1.4) with respect to θ yields

$$\partial_t \xi = (\partial A)(u)\xi, \ t > 0, \tag{1.5}$$

which is called the variation equation of (1.4). Here, we assumed that there exists the derivative $(\partial A)(u)\xi$ of A at u in the direction of ξ. If, for every such a solution $u(t, \theta)$,

$$\partial_t \|\xi\|_u \le \omega \|\xi\|_u, \tag{1.6}$$

then

$$\|\xi\|_u \le e^{\omega t} \|c'\|_c,$$
$$d(u(t, 0), u(t, 1)) \le e^{\omega t} d(x, y).$$

Setting $\|\xi\|_u = N(u, \xi)$, we have

$$\begin{aligned} \partial_t \|\xi\|_u =& \partial_t N(u, \xi) = \partial_x N(u, \xi) \partial_t u + \partial_\xi N(u, \xi) \partial_t \xi \\ =& \partial_x N(u, \xi) A u + \partial_\xi N(u, \xi)(\partial A)(u)\xi. \end{aligned}$$

Therefore, if

$$\partial_x N(x, \xi) A x + \partial_\xi N(x, \xi)(\partial A)(x)\xi \le \omega N(x, \xi)$$

for $x \in D(A)$ and $\xi \in X$, then we can expect that (1.6) holds.

Example 1.2. Assume $f : \boldsymbol{R} \to \boldsymbol{R}$ in Example 1.1 is C^1. The variation equation of $\partial_t u = f(u)$ is $\partial_t \xi = f'(u)\xi$, where ξ is the variation δu of u. Set $\|\xi\|_x = |\xi|/f(x)$. Then,

$$\begin{aligned}\partial_t \|\xi\|_u =& \partial_t \left(\frac{|\xi|}{f(u)} \right) \\ =& \frac{\mathrm{sign}(\xi)(\partial_t \xi) f(u) - |\xi| f'(u) \partial_t u}{f(u)^2} \\ =& \frac{\mathrm{sign}(\xi) f'(u) \xi f(u) - |\xi| f'(u) f(u)}{f(u)^2} = 0\end{aligned}$$

and

$$\begin{aligned}d(x,y) &= \inf \left\{ \int_0^1 \|c'(\theta)\|_{c(\theta)}\, d\theta;\ c \in \Gamma(x,y) \right\} \\ &= \int_0^1 \frac{|x-y|}{f(x+\theta(y-x))}\, d\theta = \left| \int_x^y \frac{dr}{f(r)} \right|.\end{aligned}$$

Example 1.3. Let $F \in C^1(\boldsymbol{R})$. Consider the scalar conservation law $\partial_t u + \partial_x F(u) = 0$, $-\infty < x < \infty$, $t > 0$. The variation equation for the variation $\xi = \delta u$ of u is

$$\partial_t \xi + \partial_x \left(F'(u)\xi \right) = 0$$
$$\partial_t \xi + (\partial_x F'(u))\xi + F'(u)\partial_x \xi = 0.$$

We have

$$\begin{aligned} \partial_t |\xi| =& (\partial_t \xi) \operatorname{sign}(\xi) \\ =& - \left((\partial_x F'(u))\xi + F'(u)\partial_x \xi \right) \operatorname{sign}(\xi) \\ =& - \left((\partial_x F'(u))|\xi| + F'(u)\partial_x |\xi| \right) \\ =& - \partial_x \left(F'(u)|\xi| \right), \end{aligned}$$

which implies

$$\partial_t \int_{-\infty}^{\infty} |\xi| \, dx = 0.$$

We have also

$$\begin{aligned} d(u,v) =& \inf \left\{ \int_0^1 \left(\int_{-\infty}^{\infty} |c'(\theta)| \, dx \right) \, d\theta; \;\; c \in \Gamma(u,v) \right\} \\ =& \int_{-\infty}^{\infty} |u - v| \, dx. \end{aligned}$$

Example 1.4. Consider the wave equation with a damping term:

$$\partial_t^2 w = \partial_x\left(\sigma(\partial_x w)\right) - \nu \partial_t w. \tag{1.7}$$

Here, $\nu > 0$ is a constant and σ a smooth increasing function such that

$$\sigma(0) = 0,\ \ \sigma'(r) \geq \delta_0 > 0$$

and $\|\sigma^{(j)}\|_{L^\infty} < \infty$, $j = 0, 1, 2, 3$.

Taking $u = \partial_x w$ and $v = \partial_t w$, we can rewrite (1.7) as

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x \sigma(u) - \nu v. \end{cases} \tag{1.8}$$

The variation equation for the variations $\xi = \delta u$ and $\eta = \delta v$ of u and v, respectively, is given by

$$\begin{cases} \partial_t \xi = \partial_x \eta \\ \partial_t \eta = \partial_x(\sigma'(u)\xi) - \nu\eta. \end{cases}$$

We have

$$\begin{aligned} &\sigma'(u)\partial_t|\xi|^2 + \partial_t|\eta|^2 \\ =&2\left(\partial_x\left(\sigma'(u)\xi\eta\right) - \nu|\eta|^2\right) \leq 2\partial_x\left(\sigma'(u)\xi\eta\right), \end{aligned}$$

which implies

$$\partial_t \int_{-\infty}^{\infty} \left(\sigma'(u)|\xi|^2 + |\eta|^2\right) \, dx$$

$$= \int_{-\infty}^{\infty} \left(\sigma''(u)(\partial_t u)|\xi|^2 + \sigma'(u)\partial_t|\xi|^2 + \partial_t|\eta|^2\right) \, dx$$

$$\leq \int_{-\infty}^{\infty} \sigma''(u)(\partial_t u)|\xi|^2 \, dx = \int_{-\infty}^{\infty} \sigma''(u)(\partial_x v)|\xi|^2 \, dx$$

$$\leq \delta_0^{-1} \|\sigma''\|_{L^\infty} \|\partial_x v\|_{L^\infty} \int_{-\infty}^{\infty} \sigma'(u)|\xi|^2 \, dx.$$

Therefore, if $\delta_0^{-1} \|\sigma''\|_{L^\infty} \|\partial_x v\|_{L^\infty} \leq 2\omega$, then

$$\partial_t \|(\xi, \eta)\|_{(u,v)} \leq \omega \|(\xi, \eta)\|_{(u,v)},$$

where

$$\|(\xi, \eta)\|_{(u,v)} = \left(\int_{-\infty}^{\infty} \left(\sigma'(u)|\xi|^2 + |\eta|^2\right) \, dx\right)^{1/2}.$$

Hence,

$$\partial_t d((u,v),(\hat{u},\hat{v})) \leq \omega d((u,v),(\hat{u},\hat{v})),$$

for the solutions (u,v) and $(\hat{u},\hat{v})$ of (1.8) such that

$$\delta_0^{-1}\|\sigma''\|_{L^\infty}\|\partial_x v\|_{L^\infty} \leq 2\omega,$$
$$\delta_0^{-1}\|\sigma''\|_{L^\infty}\|\partial_x \hat{v}\|_{L^\infty} \leq 2\omega.$$

Here

(1.9)

$$\begin{aligned} & d((u,v),(\hat{u},\hat{v})) \\ &= \inf\Big\{ \int_0^1 \|c'(\theta)\|_{c(\theta)}\, d\theta : \\ &\qquad c(0) = (u,v),\ c(1) = (\hat{u},\hat{v}) \Big\} \\ &= \left(\int_{-\infty}^{\infty} \left(\left(\int_{\hat{u}}^{u} \sqrt{\sigma'(r)}\, dr \right)^2 + (v-\hat{v})^2 \right) dx \right)^{1/2}. \end{aligned}$$

We see that

$$\begin{aligned} \sqrt{\delta_0 \wedge 1}\|(u,v)-(\hat{u},\hat{v})\|_{L^2\times L^2} &\leq d((u,v),(\hat{u},\hat{v})) \\ &\leq \sqrt{\|\sigma'\|_{L^\infty} \vee 1}\|(u,v)-(\hat{u},\hat{v})\|_{L^2\times L^2}. \end{aligned}$$

Let (u, v) be a solution of (1.8). Define $H(u, v)$ by

(1.10)

$$H(u,v) = \int_{-\infty}^{\infty} \left(\int_0^u \sigma(r)\, dr + \frac{1}{2} v^2 \right) dx + \frac{1}{2} \int_{-\infty}^{\infty} \left(\sigma'(u)(\partial_x u)^2 + (\nu u + \partial_x v)^2 \right) dx + \frac{1}{2} \int_{-\infty}^{\infty} \left(\sigma'(u)(\partial_x^2 u)^2 + (\nu \partial_x u + \partial_x^2 v)^2 \right) dx.$$

We see that

$$(1.11) \quad \partial_t H(u,v) \le (g(H(u,v)) - \nu) \|\partial_x u\|_{H^1}^2.$$

Here, g is a continous, increasing and non-negative function on $[0, \infty)$ such that $g(0) = 0$. Choose $r_0 > 0$ such that $g(r_0) < \nu$, and define $D \subset L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ by

$$D = \{(u,v) \in H^2(\boldsymbol{R}) \times H^2(\boldsymbol{R});\ H(u,v) \le r_0\}.$$

From (1.11), it follows that the solution $(u(t,x), v(t,x))$ of (1.8) satisfies

$$(u(t,\cdot), v(t,\cdot)) \in D,\ t \in [0, \infty),$$

if $(u(0,\cdot), v(0,\cdot)) \in D$. We can choose ω such that

$$\delta_0^{-1} \|\sigma''\|_{L^\infty} \|\partial_x v\|_{L^\infty} \le \delta_0^{-1} \|\sigma''\|_{L^\infty} \|v\|_{H^2} \le 2\omega$$

for $(u, v) \in D$.

Example 1.5. Consider the wave equation of Kirchhoff type with a damping term:

$$(1.12)\qquad \begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \beta'\left(\int_{-\infty}^{\infty} |u|^2\, dx\right)\partial_x u - \nu v. \end{cases}$$

Here, $\nu > 0$ is a constant and β a smooth increasing function such that

$$\beta(0) = 0, \quad \beta'(r) \geq \delta_0 > 0$$

and $\|\beta^{(j)}\|_{L^\infty} < \infty$, $j = 0, 1, 2$. The variation equation for the variations $\xi = \delta u$ and $\eta = \delta v$ of u and v, respectively, is given by

$$\begin{cases} \partial_t \xi = \partial_x \eta \\ \partial_t \eta = \beta'(\|u\|_{L^2}^2)\partial_x \xi - \nu\eta + 2\beta''(\|u\|_{L^2}^2)\langle u, \xi\rangle_{L^2}\partial_x u. \end{cases}$$

We have

$$\begin{aligned} &\partial_t \left(\beta'(\|u\|_{L^2}^2)\|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2\right) \\ \leq &2\beta''(\|u\|_{L^2}^2)\|(u,v)\|_{H^1\times H^1} \\ &\times (1 + 2/\delta_0)\left(\beta(\|u\|_{L^2}^2)\|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2\right). \end{aligned}$$

Therefore, if $(1+2/\delta_0)\|\beta''\|_{L^\infty}\|(u,v)\|_{H^1\times H^1} \le \omega$, then

$$\partial_t\|(\xi,\eta)\|_{(u,v)} \le \omega\|(\xi,\eta)\|_{(u,v)},$$

where

$$\|(\xi,\eta)\|_{(u,v)} = \left(\beta'(\|u\|_{L^2}^2)\|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2\right)^{1/2}.$$

The metric

$$d((u,v),(\hat{u},\hat{v}))$$
$$= \inf\left\{\int_0^1 \|c'(\theta)\|_{c(\theta)}\,d\theta : \ c(0)=(u,v),\ c(1)=(\hat{u},\hat{v})\right\}$$

cannot be written as an explicit form. Set

(1.13)

$$V((u,v),(\hat{u},\hat{v}))$$
$$= \left(\beta'(\|u\|_{L^2}^2)\|u-\hat{u}\|_{L^2}^2 + \|v-\hat{v}\|_{L^2}^2\right)^{1/2}.$$

Then, we have

$$\partial_t V((u,v),(\hat{u},\hat{v})) \le \omega V((u,v),(\hat{u},\hat{v}))$$
$$\sqrt{\delta_0\wedge 1}\|(u,v)-(\hat{u},\hat{v})\|_{L^2\times L^2} \le V((u,v),(\hat{u},\hat{v}))$$
$$\le \sqrt{\|\beta'\|_{L^\infty}\vee 1}\|(u,v)-(\hat{u},\hat{v})\|_{L^2\times L^2},$$

for the solutions (u,v) and $(\hat{u},\hat{v})$ of (1.8) such that

$$(1+2/\delta_0)\|\beta''\|_{L^\infty}\|(u,v)\|_{H^1\times H^1} \le \omega$$
$$(1+2/\delta_0)\|\beta''\|_{L^\infty}\|(\hat{u},\hat{v})\|_{H^1\times H^1} \le \omega.$$

Let (u, v) be a solution of of (1.12). Define $H(u, v)$ by

(1.14)

$$H(u,v) = \frac{1}{2}\Big(\beta(\|u\|_{L^2}^2) + \|v\|_{L^2}^2 + \beta'(\|u\|_{L^2}^2)\|\partial_x u\|_{L^2}^2 + \|\nu u + \partial_x v\|_{L^2}^2\Big).$$

We see that

(1.15)

$$\partial_t H(u,v) \leq \Big((2\|\beta''\|_{L^\infty}/\sqrt{\delta_0})H(u,v) - \nu\delta_0\Big)\|\partial_x u\|_{L^2}^2.$$

Choose $r_0 > 0$ such that $(2\|\beta''\|_{L^\infty}/\sqrt{\delta_0})r_0 \leq \nu\delta_0$, and define $D \subset L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ by

$$D = \{(u,v) \in H^1(\boldsymbol{R}) \times H^1(\boldsymbol{R});\ H(u,v) \leq r_0\}.$$

From (1.15), it follows that the solution $(u(t,x), v(t,$ of (1.12) satisfies

$$(u(t,\cdot), v(t,\cdot)) \in D,\ \ t \in [0,\infty),$$

if $(u(0,\cdot), v(0,\cdot)) \in D$. We can choose ω such that

$$(1 + 2/\delta_0)\|\beta''\|_{L^\infty}\|(u,v)\|_{H^1\times H^1} \leq \omega$$

for $(u, v) \in D$.

2.Basic properties of Infinitisimal generators.

Let D be a closed subset of X. Let $M \geq m > 0$ and $V(\cdot,\cdot)$ a nonnegative Lipschitz continuous functional on $X \times X$ such that

(2.1)

$$|V(x,y) - V(x',y')| \leq M(|x-x'| + |y-y'|),$$
$$\text{for } (x,y),(x',y') \in X \times X,$$

(2.2)

$$m|x-y| \leq V(x,y) \leq M|x-y|, \text{ for } x,y \in D.$$

Remark 2.1. The functional $V(\cdot,\cdot)$ is not assumed to satisfy the triangle inequality. But, it is a pseudo-metric on D in the sense that

$$V(x_0,x_N) \leq M|x_0 - x_N| \leq M \sum_{i=1}^{N} |x_{i-1} - x_i|$$

$$\leq \frac{M}{m} \sum_{i=1}^{N} V(x_{i-1},x_i), \text{ for } x_0, x_1, \cdots, x_N \in D.$$

Definition 2.1. Let $\{T(t)\}_{t\in[0,\infty)}$ be a semigroup on D. If, for some $\omega \in \boldsymbol{R}$,

$$V(T(t)x, T(t)y) \leq e^{\omega t} V(x,y), \ t \in [0,\infty), \ x,y \in D.$$

then the semigroup is called to be **pseudo-contractive** with respect to $V(\cdot,\cdot)$. The class of such a semigroup is denoted by $\mathcal{S}(V(\cdot,\cdot), D, \omega)$.

Define $D_+V, D^+V : (X \times X) \times (X \times X) \to \boldsymbol{R}$ by

$$\begin{aligned}
& D_+V(x,y)(\xi,\eta) \\
= & \liminf_{h \downarrow 0} (V(x+h\xi, y+h\eta) - V(x,y))/h \\
& D^+V(x,y)(\xi,\eta) \\
= & \limsup_{h \downarrow 0} (V(x+h\xi, y+h\eta) - V(x,y))/h
\end{aligned}$$

for $(x,y) \in X \times X$ and $(\xi,\eta) \in X \times X$.

Theorem 2.1. *Let* $\{T(t)\}_{t\in[0,\infty)} \in \mathcal{S}(V(\cdot,\cdot), D, \omega)$ *and* A_0 *the infinitisimal generator. Then*

$$D^+V(x,y)(A_0x, A_0y) \leq \omega V(x,y), \;\; x, \;\; y \in D(A_0),$$

$$V(T(t)x,y) - V(x,y) + \int_0^t D^+V(T(\tau)x,y)(0, A_0y)\, d\tau$$

$$\leq \omega \int_0^t V(T(\tau)x,y)\, d\tau$$

$$x \in D, \;\; y \in D(A_0), \;\; t \in [0,\infty)$$

3.Generation of Semigroups. Let $A \subset X \times X$. Consider the initial value problem for the differntial inclusion

$$\text{(DI)} \qquad \partial_t u(t) \in Au(t), \; t > 0.$$

Definition 3.1.(1) A function $u \in C^1([a,b] : X)$ is called a **classical solution** of (DI) on $[a,b]$ if it satisfies (DI) for all $t \in [a,b]$. A function $u : [a,b] \to X$ which is abosolutely continuous on $[a,b]$ and differetiable a.e. on (a,b) is called a **strong solution** of (DI) it satisfies (DI) for a.a. $t \in [a,b]$.

(2) A function $u \in C^1([0,\infty) : X)$ is called a **classical solution** of (DI) on $[0,\infty)$ if it satisfies (DI) for all $t \in [0,\infty)$. A function $u : [0,\infty) \to X$ is called a **strong solution** of (DI) on $[0,\infty)$ if $u|_{[a,b]}$ is a strong solution of (DI) on $[a,b]$ for any bounded $[a,b]$ of $[0,\infty)$.

Theorem 3.1. *Let $\omega \in \boldsymbol{R}$ and $A : D \to X$ a continuous (single-valued) operator. Assume A satifies (1) and (2) below.*

(1) *For any $x, y \in D$,*

$$D_+ V(x,y)(Ax, Ay) \le \omega V(x,y).$$

(2) *For any $\varepsilon > 0$ and $x \in D$, there exist $\lambda \in (0, \varepsilon)$ and $x_\lambda \in D$ such that*

$$|\lambda^{-1}(x_\lambda - x) - Ax| \le \varepsilon.$$

Then, there exists $\{T(t)\}_{t \in [0,\infty)} \in \mathcal{S}(V(\cdot,\cdot), D, \omega)$ such that A is the infinitesimal generator and, for $x \in D$, $u(t) = T(t)x$ is the unique classical solution of

$$\partial_t u(t) = Au(t), \ t \ge 0, \ u(0) = x.$$

Remark 3.1. The conditions (1) and (2) are necessary for A to be the infinitesimal generator of a semigroup $\{T(t)\}_{t \in [0,\infty)} \in \mathcal{S}(V(\cdot,\cdot), D, \omega)$.

Example 3.1. Consider the wave equation (1.8) with a damping term:

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x \sigma(u) - \nu v. \end{cases}$$

Let $X = L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ be the Hilbert space with norm $\|(u,v)\|_X = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$. Define a metric $V((u,v),(\hat{u},\hat{v})) = d((u,v),(\hat{u},\hat{v}))$ on X by (1.9):

$$d((u,v),(\hat{u},\hat{v})) = \left(\left(\int_{\hat{u}}^{u} \sqrt{\sigma'(r)}\, dr \right)^2 + (v - \hat{v})^2 \right)^{1/2}.$$

and a functional $H(\cdot,\cdot)$ by (1.14):

$$\begin{aligned} H(u,v) = & \int_{-\infty}^{\infty} \left(\int_0^u \sigma(r)\, dr + \frac{1}{2} v^2 \right) dx \\ & + \frac{1}{2} \int_{-\infty}^{\infty} \left(\sigma'(u)(\partial_x u)^2 + (\nu u + \partial_x v)^2 \right) dx \\ & + \frac{1}{2} \int_{-\infty}^{\infty} \left(\sigma'(u)(\partial_x^2 u)^2 + (\nu \partial_x u + \partial_x^2 v)^2 \right) dx. \end{aligned}$$

Let $r_0 > 0$ and define an operator A by

$$\begin{aligned} & A(u,v) = (\partial_x v, \partial_x \sigma(u) - \nu v), \ (u,v) \in D(A) \\ & D = D(A) \\ & = \left\{ (u,v) \in H^2(\boldsymbol{R}) \times H^2(\boldsymbol{R}); H(u,v) \le r_0 \right\}. \end{aligned}$$

If $r_0 > 0$ is suficiently small, then the following holds.

(1) $D = D(A)$ is closed in X.

(2) $A : D \to X$ is Hölder continuous:

$$\|A(u,v) - A(\hat{u},\hat{v})\|_X \leq C\|(u,v) - (\hat{u},\hat{v})\|_X^{1/2}, \quad (u,v),\ (\hat{u},\hat{v}) \in D.$$

(3) For some $\omega \in \boldsymbol{R}$,

$$D_+V\left((u,v),(\hat{u},\hat{v})\right)\left(A(u,v),A(\hat{u},\hat{v})\right) \leq \omega V\left((u,v),(\hat{u},\hat{v})\right), \quad (u,v),(\hat{u},\hat{v}) \in D.$$

(4) For any $(u_0,v_0) \in D$, there exists $\lambda_0 > 0$ such that, for any $\lambda \in (0,\lambda_0]$, there exists $(u_\lambda,v_\lambda) \in D$ satisfying

$$\begin{cases} \lambda^{-1}(u_\lambda - u_0) = \partial_x v_\lambda \\ \lambda^{-1}(v_\lambda - v_0) = \sigma''(u_0)\partial_x u_\lambda - \nu v_\lambda \end{cases}$$

and

$$\lim_{\lambda\downarrow 0}\left\|\lambda^{-1}\left((u_\lambda,v_\lambda) - (u_0,v_0)\right) - A(u_0,v_0)\right\|_X = 0.$$

(5) There exists a semigroup $\{T(t)\}_{t\in[0,\infty)} \in \mathcal{S}(V(\cdot,\cdot), D, \omega)$ such that, for $(u_0, v_0) \in D$, $(u(t,\cdot), v(t,\cdot)) = T(t)(u_0, v_0)$ is the unique classical solution of

$$\partial_t(u(t,\cdot), v(t,\cdot)) = A(u(t,\cdot), v(t,\cdot)),\ t \in [0,\infty)$$
$$(u(t,\cdot), v(t,\cdot))|_{t=0} = (u_0, v_0).$$

Definition 3.2. Let $\varepsilon > 0$. A function $u : [a, b] \to X$ is called an **ε-approximate solution** of (DI) on $[a, b]$, if there exist a partition $a = t_0 < t_1 < \cdots < t_N = b$ of $[a, b]$ and a sequence $(x_i, \xi_i) \in A$, $i = 1, 2, \cdots, N$ such that

$$(\varepsilon 1) \quad u(t) = x_i, \ t \in (t_{i-1}, t_i], \ i = 1, 2, \cdots, N,$$

$$(\varepsilon 2) \quad \sum_{i=1}^{N} |x_i - x_{i-1} - (t_i - t_{i-1})\xi_i| \le \varepsilon,$$

$$(\varepsilon 3) \quad \max_{i=1,2,\cdots,N} (t_i - t_{i-1}) \le \varepsilon,$$

where, $x_0 = u(a)$.

Definition 3.3. (1) A function $u \in C([a, b] : X)$ is called a **mild solution** of (DI) on $[a, b]$, if, for any $\varepsilon > 0$, there exists an ε-approximate solution u_ε of (DI) on $[a, b(\varepsilon)]$ such that

$$\sup_{t \in [a,b]} |u(t) - u_\varepsilon(t)| \le \varepsilon, \ b \le b(\varepsilon) < b + \varepsilon.$$

(2) A function $u \in C([0, \infty) : X)$ is called a **mild solution** of (DI) on $[0, \infty)$, if $u |_{[a,b]}$ is a mild solution of (DI) on $[a, b]$ for any bounded $[a, b] \subset [0, \infty)$.

Theorem 3.2. *Let* $\omega \in \boldsymbol{R}$ *and* $A \subset X \times X$ *such that* $D = \overline{D(A)}$. *Assume* A *satisfies* (1) *and* (2) *below.*

(1) *For any* $(x_0, y_0) \in D \times D$, *there exists a neighborhood* $W(\subset X \times X)$ *of* (x_0, y_0) *such that*

$$\limsup_{\lambda \downarrow 0,\, \mu \downarrow 0} \Bigg(\sup \Big\{ \left(V(x,y) - V(x - \lambda\xi, y)\right)/\lambda \\ + \left(V(x,y) - V(x, y - \mu\eta)\right)/\mu - \omega V(x,y); \\ (x,y) \in W,\ (x,\xi),\ (y,\eta) \in A \Big\} \Bigg) \leq 0.$$

(2) *For any* $\varepsilon > 0$ *and* $x \in D$, *there exist* $\lambda \in (0, \varepsilon)$ *and* $(x_\lambda, \xi_\lambda) \in A$ *such that*

$$|x_\lambda - x| \leq \varepsilon, \quad |\lambda^{-1}(x_\lambda - x) - \xi_\lambda| \leq \varepsilon.$$

Then, there exists $\{T(t)\}_{t \in [0,\infty)} \in \mathcal{S}(V(\cdot,\cdot), D, \omega)$ *such that, for any* $x \in D$, $u(\cdot) = T(\cdot)x$ *is the unique mild solution of*

$$\partial_t u(t) \in Au(t),\ t \geq 0,\ u(0) = x.$$

Remark 3.5. If $V : X \times X \to [0, \infty)$ is convex, then the condition (1) is equivalent to the next.

$$0 \le D_+ V(x,y)(-\xi, 0) + D_+ V(x,y)(0, -\eta) \\ +\omega V(x,y), \quad (x,\xi),\ (y,\eta) \in A.$$

Remark 3.6. If $D = D(A)$ and $A : D(A) \to X$ is continuous, then the condition (2) is equivalent to (2) in Theorem 3.1.

Proof.

(1) If u and $\hat{u}$ are mild solutions of (DI) on $[a,b]$, then

$$V(u(t), \hat{u}(t)) \le e^{\omega(t-a)} V(u(a), \hat{u}(a)), \quad t \in [a,b].$$

(2) If, for any $x \in D$, there exist $\tau_x > 0$ and a mild solution of (DI) on $[0, \tau_x]$ satisfying $u(0) = x$, then, for any $x \in D$, there exists a mild solution of (DI) on $[0, \infty)$ satisfying $u(0) = x$.

(3) Let $[a,b] \subset [0,\infty)$ and, for any $\varepsilon > 0$, let u_ε be an ε - approximate solution of (DI) on $[a_\varepsilon, b_\varepsilon]$ such that $[a,b] \subset [a_\varepsilon, b_\varepsilon]$. If $\lim_{\varepsilon\downarrow 0} u_\varepsilon(a) = x \in D$ and

$$(3.1) \qquad \lim_{\substack{\varepsilon\downarrow 0,\ |t-s|\to 0 \\ t,\ s\in[a,b]}} |u_\varepsilon(t) - u_\varepsilon(s)| = 0,$$

then there exists a mild solution u of (DI) on $[a,b]$ such that

$$\lim_{\varepsilon\downarrow 0} \left(\sup_{t\in[a,b]} |u(t) - u_\varepsilon(t)| \right) = 0.$$

(4) For any $x \in D$, there exists $\tau_x > 0$ such that, for any $\varepsilon > 0$, there exist $\tau_\varepsilon \geq \tau_x$ and ε - approximate solution u_ε of (DI) on $[0, \tau_\varepsilon]$ satisfying $u_\varepsilon(0) = x$ and (3.1) with $[a,b] = [0, \tau_x]$.

Q.E.D.

Example 3.2. Consider the wave equation (1.12) of Kirchhoff type with a damping term:

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \beta'\left(\int_{-\infty}^{\infty} |u|^2\, dx\right) \partial_x u - \nu v. \end{cases}$$

Let $X = L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ be the Hilbert Space with norm $\|(u, v)\|_X = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$. Define a functional $V(\cdot, \cdot)$ on $X \times X$ by (1.13):

$$V((u, v), (\hat{u}, \hat{v})) \\ = \left(\beta'(\|u\|_{L^2}^2)\|u - \hat{u}\|_{L^2}^2 + \|v - \hat{v}\|_{L^2}^2\right)^{1/2}$$

and a functional $H(\cdot, \cdot)$ by (1.14):

$$H(u, v) = \frac{1}{2}\Big(\beta(\|u\|_{L^2}^2) + \|v\|_{L^2}^2 \\ + \beta'(\|u\|_{L^2}^2)\|\partial_x u\|_{L^2}^2 + \|\nu u + \partial_x v\|_{L^2}^2\Big).$$

We see the functional $V(\cdot, \cdot)$ satisfies the conditions (2.1) and (2.2):

$$|V(x, y) - V(x', y')| \leq M(|x - x'| + |y - y'|), \\ \text{for } (x, y), (x', y') \in X \times X, \\ m|x - y| \leq V(x, y) \leq M|x - y|, \text{ for } x, y \in D.$$

Let $r_0 > 0$ and define an operator A by

(3.2)
$$A(u,v) = (\partial_x v, \beta'(\|u\|^2_{L^2})\partial_x u - \nu v), \ (u,v) \in D(A)$$
$$D = D(A) =$$
(3.3)
$$\left\{(u,v) \in H^1(\boldsymbol{R}) \times H^1(\boldsymbol{R}) : H(u,v) \le r_0\right\}.$$

If $r_0 > 0$ is suficiently small, then the following holds.

(1) $D = D(A)$ is closed in X.

(2) For some $\omega \in \boldsymbol{R}$,

$$\limsup_{\lambda\downarrow 0,\, \mu\downarrow 0}\Bigg(\sup\Big\{-\omega V((u,v),(\hat{u},\hat{v}))$$
$$+ \left(V((u,v),(\hat{u},\hat{v})) - V((u,v) - \lambda A(u,v),(\hat{u},\hat{v}))\right)/\lambda$$
$$+ \left(V((u,v),(\hat{u},\hat{v})) - V((u,v),(\hat{u},\hat{v}) - \mu A(\hat{u},\hat{v}))\right)/\mu;$$
$$(u,v),\ (\hat{u},\hat{v}) \in D(A)\Big\}\Bigg) \le 0.$$

(3) For any $(u_0, v_0) \in D$, there exists $\lambda_0 > 0$ such that, for any $\lambda \in (0, \lambda_0]$, there exist

$(u_\lambda, v_\lambda) \in D$ satisfying

$$\begin{cases} u_\lambda - u_0 = \lambda \partial_x v_\lambda \\ v_\lambda - v_0 = \lambda \beta'(\|u_0\|_{L^2}^2)\partial_x u_\lambda - \lambda \nu v_\lambda \end{cases}$$

and

$$\lim_{\lambda \downarrow 0} \|(u_\lambda, v_\lambda) - (u_0, v_0)\|_X = 0,$$

$$\lim_{\lambda \downarrow 0} \left\| \lambda^{-1} \left((u_\lambda, v_\lambda) - (u_0, v_0) \right) - A(u_\lambda, v_\lambda) \right\|_X = 0.$$

(4) There exists a semigroup $\{T(t)\}_{t\in[0,\infty)} \in \mathcal{S}(V(\cdot,\cdot), D, \omega)$ such that, for $(u_0, v_0) \in D$, $(u(t,\cdot), v(t,\cdot)) = T(t)(u_0, v_0)$ is the unique mild solution of

$$\partial t(u(t,\cdot), v(t,\cdot)) = A(u(t,\cdot), v(t,\cdot)), \ t \in [0,\infty)$$

$$(u(t,\cdot), v(t,\cdot))|_{t=0} = (u_0, v_0). \tag{3.4}$$

Remark 3.8. In fact, A is the infinitisimal generator of the semigroup $\{T(t)\}_{t\in[0,\infty)}$ and, for any $(u_0, v_0) \in D$, $(u(t,\cdot), v(t,\cdot)) = T(t)(u_0, v_0)$ is the unique strong solution of (3.4).

4.Approximation by Discrete Semigroups. Let $\{T(t)\}_{t\in[0,\infty)}$ be a semigroup of Lipschitz operators on D and, for $h \in (0, h_0]$, C_h a Lipschitz operator on D_h into itself, where $D \subset D_h \subset X$.

Theorem 4.1. *Assume $\{C_h\}_{h\in(0,h_0]}$ satisfies (S) and (C) below.*

(S) *For $\tau > 0$, there exists $M_\tau \geq 1$ such that*

$$|C_h^n x - C_h^n y| \leq M_\tau |x - y|$$

for x, $y \in D_h$, $h \in (0, h_0]$ and $n = 1, 2, \ldots$ such that $nh \in (0, \tau]$.

(C) *There exists a dense subset D_0 of D such that*

$$\lim_{h\downarrow 0} \int_0^\tau \frac{1}{h} |T(t+h)x - C_h T(t)x| \, dt = 0$$

for $x \in D_0$ and $\tau > 0$.

Then

$$\lim_{h\downarrow 0} \left(\sup_{t\in[0,\tau]} |T(t)x - C_h^{[t/h]} x| \right) = 0$$

for $x \in D$ and $\tau > 0$, where $[a]$ denotes the greatest integer not greater than a.

Lemma 4.1. *Asuume* $\{C_h\}_{h\in(0,h_0]}$ *satisfies* (S). *Let* $\tau > 0$ *and* $u \in C([0,\tau] : X)$ *such that* $u([0,\tau]) \subset D$. *Then*

$$\begin{aligned}
&|u(t) - C_h^{[t/h]}u(0)| \\
\le &\frac{M_\tau}{h}\int_0^\tau |u(s+h) - C_h u(s)|\, ds + (1 + M_\tau) \\
&\times \sup\Big\{|u(s) - u(\hat{s})|;\ s,\ \hat{s} \in [0,\tau + h_0],\ |s - \hat{s}| \le h\Big\}
\end{aligned}$$

for $t \in [0,\tau]$ *and* $h \in (0,h_0]$.

Theorem 4.2. *Let* $u \in C([0,\tau] : X)$ *be Lipschitz continous and* $u([0,\tau]) \subset \hat{D}$. *If* $s \to \lim_{h\downarrow 0}|T(h)u(s)-u(s)|/h$ *is essentially bounded on* $[0,\tau]$ *and*

$$\liminf_{h\downarrow 0} \frac{1}{h}|u(s+h) - T(h)u(s)| = 0$$

for a.a. $s \in (0,\tau)$, *then* $u(t) = T(t)u(0)$ *for all* $t \in [0,\tau]$.

Theorem 4.3. $\{C_h\}_{h\in(0,h_0]}$ *satisfies* (S) *if and only if there exist* $M \geq 1$, $\omega \geq 0$ *and* $V_h : D_h \times D_h \to [0,\infty)$, $h \in (0,h_0]$, *such that*

$$V_h(C_h x, C_h y) \leq e^{\omega h} V_h(x,y)$$
$$|x-y| \leq V_h(x,y) \leq M|x-y|$$

for $h \in (0,h_0]$ *and* $x,\ y \in D_h$.

Theorem 4.4. *Let* A_0 *be the infinitesimal generator of* $\{T(t)\}_{t\in[0,\infty)}$. *Assume that there exists a dense subset* D_0 *of* D *such that* $u(t) = T(t)x$ *is the strong solution of*

$$\partial_t(t) = A_0 u(t), \quad t > 0; \quad u(0) = x.$$

for any $x \in D_0$. *Then the next* (C″) *implies* (C).

(C″) *For* $x \in D_0$ *and* $\tau > 0$,

$$\lim_{h\downarrow 0}\left(\int_0^\tau |A_h T(t)x - A_0 T(t)x|\,dt\right) = 0,$$

where $A_h x = h^{-1}(C_h x - x)$ *for* $x \in D_h$.

Example 4.1. Consider the initial value problem for (1.12):

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \beta'\left(\int_{-\infty}^{\infty} |u|^2\, dx\right) \partial_x u - \nu v. \end{cases}$$

and the difference scheme of Friedrichs-Lax's type for the initial value problem:

$$\frac{1}{h}\left(u_h^n(x) - \frac{1}{2}(u_h^{n-1}(x+k) + u_h^{n-1}(x-k))\right)$$
$$= \frac{1}{2k}\left(v_h^{n-1}(x+k) - v_h^{n-1}(x-k)\right),$$
$$n = 1,\ 2,\ \ldots, \quad -\infty < x < \infty$$
$$\frac{1}{h}\left(v_h^n(x) - \frac{1}{2}(v_h^{n-1}(x+k) + v_h^{n-1}(x-k))\right)$$
$$= \beta'(\int_{-\infty}^{\infty} |u_h^n(x)|^2\, dx) \frac{1}{2k}\left(u_h^{n-1}(x+k) - u_h^{n-1}(x-k)\right)$$
$$- \frac{\nu}{2}\left(v_h^{n-1}(x+k) + v_h^{n-1}(x-k)\right),$$
$$n = 1,\ 2,\ \ldots, \quad -\infty < x < \infty,$$
$$u_h^0(x) = u_0(x), \;\; v_h^0(x) = v_0(x), \qquad -\infty < x < \infty,$$

where $h > 0$ and $k > 0$ represent the differencing in time and in space, respectively.

Let $E_0 > 0$, the ratio $r = h/k$ kept as a constant as $h \downarrow 0$, and $r\sqrt{\beta'(E_0/\delta_0)} < 1$. Let $X = L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R})$ and define $V(\cdot,\cdot)$ by (1.13), A with $D = D(A)$ by (3.2) and (3.3), as in Example 3.2. Set $E : X \to [0,\infty)$ by

$$E(u,v) = \frac{1}{2}\beta\left(\|u\|_{L^2}^2\right) + \frac{1}{2}\|v\|_{L^2}^2$$

and, for $h > 0$, $H_h : X \to [0,\infty)$ by

$$\begin{aligned} H_h(u,v) &= E(u,v) \\ &+ \frac{1}{2}\beta'\left(\|u\|_{L^2}^2\right)\|\frac{1}{2k}(u(\cdot+k) - u(\cdot-k))\|_{L^2}^2 \\ &+ \frac{1}{2}\Big\|\frac{1}{2k}(v(\cdot+k) - v(\cdot-k)) \\ &\qquad + \frac{\nu}{2}(u(\cdot+k) + u(\cdot-k))\Big\|_{L^2}^2. \end{aligned}$$

Let $r_h > 0$ and define D_h by

$$\begin{aligned} D_h = \Big\{ &(u,v) \in L^2(\boldsymbol{R}) \times L^2(\boldsymbol{R}); \\ &E(u,v) \le E_0,\ H_h(u,v) \le r_h \Big\}. \end{aligned}$$

and $C_h : D_h \to X$ by

$$C_h(u,v) = (u_h, v_h) \quad (u,v) \in D_h,\ h \in (0,h_0]$$

where

$$
\begin{aligned}
u_h(x) &= \frac{1}{2}(u(x+k) + u(x-k)) \\
&\quad + \frac{h}{2k}(v(x+k) - v(x-k)) \\
v_h(x) &= \frac{1}{2}(v(x+k) + v(x-k)) \\
&\quad + \beta'\left(\int_{-\infty}^{\infty} |u_h(x)|^2\, dx\right)\frac{h}{2k}(u(x+k) - u(x-k)) \\
&\quad - \frac{\nu h}{2}(v(x+k) + v(x-k))
\end{aligned}
$$

for $h \in (0, h_0]$.

Let $\{T(t)\}_{t\in[0,\infty)}$ be the semigroup in Example 3.2. If $r_0 > 0$, $h_0 > 0$, and $r_h > 0$, $h \in (0, h_0]$, are suitably chosen, then the following holds.

(1) For each $h \in (0, h_0]$, $D = D(A) \subset D_h$ and C_h maps D_h into itself.

(2) There exists $\omega \geq 0$ such that

$$V(C_h(u,v), C_h(\hat{u},\hat{v})) \leq e^{h\omega} V((u,v),(\hat{u},\hat{v}))$$

for (u,v), $(\hat{u},\hat{v}) \in D_h$ and $h \in (0, h_0]$.

(3) For $(u_0, v_0) \in D$ and $\tau > 0$,

$$\lim_{h\downarrow 0} \int_0^\tau \|A_h T(t)(u_0,v_0) - A_0 T(t)(u_0,v_0)\|_X \, dt = 0$$

where A_0 is the infinitesimal generator of the semigroup $\{T(t)\}_{t\in[0,\infty)}$ and

$$A_h(u,v) = \frac{1}{h}(C_h(u,v) - (u,v))$$
$$(u,v) \in D_h, \ \ h \in (0, h_0].$$

(4)

$$\lim_{h\downarrow 0} \left(\sup_{t\in[0,\tau]} \|T(t)(u_0,v_0) - C_h^{[t/h]}(u_0,v_0)\|_X \right) = 0$$

for $(u_0, v_0) \in D$ and $\tau > 0$.