STUDENTIZED ROBUST STATISTICS IN TWO-WAY MANOVA WITH INTERACTION

Taka-aki Shiraishi
Department of Mathematical Sciences
Yokohama City University
22-2 Seto Kanazawa-Ku Yokohama 236-0027, Japan

Abstract

Multiresponse experiments in two-way layouts with interaction, having equal number of observations per cell, are considered. Statistical procedures of the test and estimation, based on studentized robust statistics, for location parameters in the models are proposed. Large sample properties of their procedures as the cell size tends to infinity are investigated. Although Fisher's consistency is assumed in the theory of M-estimators, it is not needed in this paper. For the univariate case, it is found that the asymptotic relative efficiencies (ARE's) of the proposed procedures relative to classical procedures agree with the classical ARE-results of Huber's one sample M-estimator relative to the sample mean. By simulation studies, it can be seen that the proposed estimators are more efficient than least squares estimators except for the case where the underlying distribution is normal.

We consider two-way MANOVA model with interaction, having equal number of observations per cell. For the two-way model, the k-th observation $\boldsymbol{X}_{ijk} = (X_{ijk}^{(1)}, \cdots, X_{ijk}^{(p)})'$ in the i-th level of the first factor and j-th level of the second factor is expressed as

$$\boldsymbol{X}_{ijk} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\beta}_j + (\boldsymbol{\alpha\beta})_{ij} + \boldsymbol{e}_{ijk}, \qquad (i = 1, \cdots, I, j = 1, \cdots, J, k = 1, \cdots, n) \tag{1.1}$$

where $\sum_{i=1}^{I} \boldsymbol{\alpha}_i = \sum_{j=1}^{J} \boldsymbol{\beta}_j = \boldsymbol{0}$ and $\sum_{i=1}^{I} (\boldsymbol{\alpha\beta})_{ij} = \sum_{j=1}^{J} (\boldsymbol{\alpha\beta})_{ij} = \boldsymbol{0}$ for all i, j's. In (1.1), $\boldsymbol{\mu}$ is the overall mean response, $\boldsymbol{\alpha}_i$ is the effect of the i-th level of the first factor, $\boldsymbol{\beta}_j$ is the effect of the j-th level of the second factor, $(\boldsymbol{\alpha\beta})_{ij}$ is the interaction between the i-th level of the first factor and the j-th level of the second factor and $\boldsymbol{e}_{ijk}$ is the error term with $E(\boldsymbol{e}_{ijk}) = \boldsymbol{0}$. The terms $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_j$ are also called main effects. It is assumed that $\boldsymbol{e}_{ijk}$'s are independent and identically distributed with continuous distribution function $F(x^{(1)}/\sigma^{(1)}, \cdots, x^{(p)}/\sigma^{(p)})$, where $V(e_{ijk}^{(\ell)}) = \{\sigma^{(\ell)}\}^2$ and $e_{ijk}^{(\ell)}$ is the ℓ-th element of $\boldsymbol{e}_{ijk}$. For the respective parameters, the null hypotheses of interest and the alternatives are respectively

$$H;\ (\boldsymbol{\alpha\beta})_{ij} = \boldsymbol{0} \text{ for } i = 1, \cdots, I \text{ and } j = 1, \cdots, J \text{ v.s. } A;\ (\boldsymbol{\alpha\beta})_{ij} \neq \boldsymbol{0} \text{ for some } (i, j).$$

$$H';\ \boldsymbol{\alpha}_i = \boldsymbol{0} \text{ for } i = 1, \cdots, I \text{ v.s. } A';\ \boldsymbol{\alpha}_i \neq \boldsymbol{0} \text{ for some } i.$$

$$H^*;\ \boldsymbol{\beta}_j = \boldsymbol{0} \text{ for } j = 1, \cdots, J \text{ v.s. } A^*;\ \boldsymbol{\beta}_j \neq \boldsymbol{0} \text{ for some } j.$$

We propose test procedures based on studentized robust statistics for these hypotheses in the model (1.1). We derive their asymptotic properties as cell size n tends to infinity. We also propose robust estimators studentized by scale-estimators for the interactions, main effects and overall mean response, and derive their asymptotic normality.

Multivariate M-tests and M-estimators based on studentized robust statistics were discussed by Singer and Sen (1985) for full rank linear models . Further discussions of studentized M-procedures were done by Koenker and Portnoy (1990) and Jurečková and Sen (1995). The linear models do not include our model (1.1), which is not of full rank. In all of those theoretical discussions, Fisher's consistency: $\int \psi(x)dF(x) = 0$ was needed. Shiraishi (1990) discussed multivariate M-tests without assuming Fisher's consistency. However Shiraishi's test statistics are not studentized. In practically applied model assumptions, the scale-parameter of the underlying distribution is unknown and Fisher's consistency does not hold. We need to construct flexible statistical procedures. For the model (1.1), we propose studentized robust tests and computable robust estimators. The asymptotic noncentral χ^2-distributions for the test statistics and asymptotic normality for robust estimators are derived, assuming only the finiteness of Fisher's informations. Fisher's consistency : $\int \psi_\ell(x^{(\ell)})dF_\ell(x^{(\ell)}) = 0$ is not assumed. Shiraishi (1991) discussed rank procedures in the model (1.1) and showed that, for the univariate cases, the Pitman asymptotic relative efficiency (ARE) of the rank procedures relative to the normal theory methods agrees with the of the proposed tests (proposed estimators) relative to the F-tests (least squares estimators (LSE's)) agrees with the ARE of the Huber's (1964) M-estimators relative to the one-sample sample mean. By simulation study, even for the small cell sizes, it can be seen that the proposed estimators are more efficient than least squares estimators except for the case where the underlying distribution is normal.

In Section 2, studentized robust procedures are proposed. In Section 3, asymptotic linearity is studied for robust statistics. In Section 4, the asymptotic χ^2-distributions of the test statistics for respective parameters are derived. In Sections 5, asymptotic multivariate normality for the robust estimators are studied. In Section 6, the asymptotic efficiencies of the proposed procedures relative to parametric procedures and robustness due to Huber (1964) are studied. In Section 7, simulation study for small cell sizes is done.

2 Robust Procedures

Because of the construction of robust statistics and because of comparison with robust procedures, we put LSE's of $\boldsymbol{\mu}$, $\boldsymbol{\alpha}_i$, $\boldsymbol{\beta}_j$ and $(\boldsymbol{\alpha\beta})_{ij}$ by $\tilde{\boldsymbol{\mu}} = \bar{\boldsymbol{X}}_{\cdots}$, $\tilde{\boldsymbol{\alpha}}_i = \bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdots}$, $\tilde{\boldsymbol{\beta}}_j = \bar{\boldsymbol{X}}_{\cdot j\cdot} - \bar{\boldsymbol{X}}_{\cdots}$, and $\tilde{(\boldsymbol{\alpha\beta})}_{ij} = \bar{\boldsymbol{X}}_{ij\cdot} - \bar{\boldsymbol{X}}_{i\cdot\cdot} - \bar{\boldsymbol{X}}_{\cdot j\cdot} + \bar{\boldsymbol{X}}_{\cdots}$, respectively, where $\bar{\boldsymbol{X}}_{\cdots} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/N$, $\bar{\boldsymbol{X}}_{i\cdot\cdot} = \sum_{j=1}^{J}\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/(Jn)$, $\bar{\boldsymbol{X}}_{\cdot j\cdot} = \sum_{i=1}^{I}\sum_{k=1}^{n}\boldsymbol{X}_{ijk}/(In)$, $\bar{\boldsymbol{X}}_{ij\cdot} = \sum_{k=1}^{n}\boldsymbol{X}_{ijk}/n$, and $N = IJn$. The ℓ-th elements of $\tilde{\boldsymbol{\mu}}$, $\tilde{\boldsymbol{\alpha}}_i$, $\tilde{\boldsymbol{\beta}}_j$ and $\tilde{(\boldsymbol{\alpha\beta})}_{ij}$ are respectively defined by $\tilde{\mu}^{(\ell)}$, $\tilde{\alpha}_i^{(\ell)}$ $\tilde{\beta}_j^{(\ell)}$ and $\tilde{(\alpha\beta)}_{ij}^{(\ell)}$. For $p \times (IJ)$ matrices $\boldsymbol{T} = (\boldsymbol{t}_{11}, \cdots, \boldsymbol{t}_{1J}, \boldsymbol{t}_{21}, \cdots, \boldsymbol{t}_{IJ}) = (t_{ij}^{(\ell)})$, $\boldsymbol{\Theta} = (\boldsymbol{\theta}_{11}, \cdots, \boldsymbol{\theta}_{1J}, \boldsymbol{\theta}_{21}, \cdots, \boldsymbol{\theta}_{IJ}) = (\theta_{ij}^{(\ell)})$ and for $p \times J$ matrix $\boldsymbol{\Theta}^* = (\boldsymbol{\theta}_1^*, \cdots, \boldsymbol{\theta}_J^*) = (\theta_j^{*(\ell)})$ and p dimensional column vectors $\boldsymbol{t} = (t^{(1)}, \cdots, t^{(p)})'$, $\boldsymbol{\theta} = (\theta^{(1)}, \cdots, \theta^{(p)})'$ and $\boldsymbol{s} = (s^{(1)}, \cdots, s^{(p)})'$, let us define aligned observations by $X_{ijk}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) = \{X_{ijk}^{(\ell)} - t_{ij}^{(\ell)} - \theta_{ij}^{(\ell)}\}/s^{(\ell)}$, $X_{ijk}^{*(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}^*, \boldsymbol{s}) = \{X_{ijk}^{(\ell)} - t_{ij}^{(\ell)} - \theta_j^{*(\ell)}\}/s^{(\ell)}$, $X_{ijk}^{\sharp(\ell)}(\boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{s}) = \{X_{ijk}^{(\ell)} - t^{(\ell)} - \theta^{(\ell)}\}/s^{(\ell)}$ and $X_{ijk}^{\natural(\ell)}(\boldsymbol{T}, \boldsymbol{\theta}, \boldsymbol{s}) = \{X_{ijk}^{(\ell)} - t_{ij}^{(\ell)} - \theta^{(\ell)}\}/s^{(\ell)}$. Then for function $\psi_\ell(x)$ defined on R^1 and for $p \times (IJ)$ matrices $\boldsymbol{T}$, $\boldsymbol{\Theta}$, $p \times J$ matrices $\boldsymbol{\Theta}^*$ and p dimensional column vectors $\boldsymbol{\theta}$, $\boldsymbol{t}$ and $\boldsymbol{s}$, let us put

$$\begin{aligned} & M_{ij}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) \\ (2.1) \qquad & = \sum_{k=1}^{n}\{\psi_\ell(X_{ijk}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})) - \bar{\psi}_\ell(X_{.jk}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})) - \bar{\psi}_\ell(X_{i.k}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})) + \bar{\psi}_\ell(X_{\ldots}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}))\}/\sqrt{n}, \end{aligned}$$

$$(2.2) \qquad M_j^{*(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}^*, \boldsymbol{s}) = \sum_{i=1}^{I}\sum_{k=1}^{n}\{\psi_\ell(X_{ijk}^{*(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}^*, \boldsymbol{s})) - \bar{\psi}_\ell(X_{\ldots}^{*(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}^*, \boldsymbol{s}))\}/\sqrt{n},$$

$$(2.3) \qquad M_{ijj'}^{\sharp(\ell)}(\boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{s}) = \sum_{k=1}^{n}\{\psi_\ell(X_{ijk}^{\sharp(\ell)}(\boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{s})) - \psi_\ell(X_{ij'k}^{\sharp(\ell)}(\boldsymbol{t}, -\boldsymbol{\theta}, \boldsymbol{s}))\}/\sqrt{n},$$

$$M^{\natural(\ell)}(\boldsymbol{T},\boldsymbol{\theta},\boldsymbol{s}) = \sum_{I=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\psi_\ell(X_{ijk}^{\natural(\ell)}(\boldsymbol{T},\boldsymbol{\theta},\boldsymbol{s}))/\sqrt{n}, \tag{2.4}$$

respectively, where $\bar{\psi}_\ell(X_{.jk}^{(\ell)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s})) = \sum_{i=1}^{I}\psi_\ell(X_{ijk}^{(\ell)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}))/I$,
$\bar{\psi}_\ell(X_{i.k}^{(\ell)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s})) = \sum_{j=1}^{J}\psi_\ell(X_{ijk}^{(\ell)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}))/J$, $\bar{\psi}_\ell(X_{\cdots}^{(\ell)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s})) = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\psi_\ell(X_{ijk}^{(\ell)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}))/N$ and $\bar{\psi}_\ell(X_{\cdots}^{*(\ell)}(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s}))$ is also the sample mean about the dots $\cdots$.

2.1 Tests

Let us put $p\times(IJ)$ matrix

$$\boldsymbol{M}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}) = (\boldsymbol{M}_{11}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}),\cdots,\boldsymbol{M}_{1J}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}),\boldsymbol{M}_{21}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}),\cdots,\boldsymbol{M}_{IJ}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s})), \tag{2.5}$$

where $\boldsymbol{M}_{ij}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}) = (M_{ij}^{(1)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s})),\cdots,M_{ij}^{(p)}(\boldsymbol{T},\boldsymbol{\Theta},\boldsymbol{s}))'$. Then we propose to reject H when the following statistic is too large.

$$MT = \mathrm{vec}(\boldsymbol{M}(\tilde{\boldsymbol{T}}_n,O_{p\times IJ},\tilde{\boldsymbol{s}}_n))'(E_{IJ}\otimes\hat{\Gamma}^{-1})\mathrm{vec}(\boldsymbol{M}(\tilde{\boldsymbol{T}}_n,O_{p\times IJ},\tilde{\boldsymbol{s}}_n)),$$

where $\mathrm{vec}(Z) = (\boldsymbol{z}_1',\cdots,\boldsymbol{z}_K')'$ for $p\times K$ matrix $Z = (\boldsymbol{z}_1,\cdots,\boldsymbol{z}_K)$, E_m denotes the m-dimensional identity matrix, $\hat{\Gamma} = (\hat{\gamma}_{\ell\ell'})_{\ell,\ell'=1,\cdots,p}$, and

$$\begin{aligned}\hat{\gamma}_{\ell\ell'} &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\{\psi_\ell(X_{ijk}^{(\ell)}(\tilde{\boldsymbol{T}}_n,O_{p\times IJ},\tilde{\boldsymbol{s}}_n)) - \bar{\psi}_\ell(X_{\cdots}^{(\ell)}(\tilde{\boldsymbol{T}}_n,O_{p\times IJ},\tilde{\boldsymbol{s}}_n))\}\\ &\quad\{\psi_{\ell'}(X_{ijk}^{(\ell')}(\tilde{\boldsymbol{T}}_n,O_{p\times IJ},\tilde{\boldsymbol{s}}_n)) - \bar{\psi}_{\ell'}(X_{\cdots}^{(\ell')}(\tilde{\boldsymbol{T}}_n,O_{p\times IJ},\tilde{\boldsymbol{s}}_n))\}/\{IJ(n-1)\},\end{aligned} \tag{2.6}$$

$\tilde{\boldsymbol{T}}_n = (\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_1+\tilde{\boldsymbol{\beta}}_1,\cdots,\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_1+\tilde{\boldsymbol{\beta}}_J,\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_2+\tilde{\boldsymbol{\beta}}_1,\cdots,\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_I+\tilde{\boldsymbol{\beta}}_J)$, $\tilde{\boldsymbol{s}}_n = (\tilde{s}^{(1)},\cdots,\tilde{s}^{(p)})'$ and

$$\tilde{s}^{(\ell)} = \sqrt{\frac{\pi}{2}}\cdot\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}|X_{ijk}^{(\ell)} - \tilde{\mu}^{(\ell)} - \tilde{\alpha}_i^{(\ell)} - \tilde{\beta}_j^{(\ell)} - (\tilde{\alpha\beta})_{ij}^{(\ell)}|/N:\ \text{mean absolute deviation.}$$

Next let us put $p\times J$ matrix

$$\boldsymbol{M}^*(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s}) = (\boldsymbol{M}_1^*(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s}),\cdots,\boldsymbol{M}_J^*(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s})), \tag{2.7}$$

where $\boldsymbol{M}_j^*(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s}) = (M_j^{*(1)}(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s})),\cdots,M_j^{*(p)}(\boldsymbol{T},\boldsymbol{\Theta}^*,\boldsymbol{s}))'$. Then we propose to reject H^* when the following statistic is too large.

$$MT^* = \mathrm{vec}(\boldsymbol{M}^*(\tilde{\boldsymbol{T}}_n^*,O_{p\times J},\tilde{\boldsymbol{s}}_n))'(E_J\otimes\hat{\Gamma}^{*-1})\mathrm{vec}(\boldsymbol{M}^*(\tilde{\boldsymbol{T}}_n^*,O_{p\times J},\tilde{\boldsymbol{s}}_n))/I,$$

where $\hat{\Gamma}^* = (\hat{\gamma}_{\ell\ell'}^*)_{\ell,\ell'=1,\cdots,p}$, and

$$\begin{aligned}\hat{\gamma}_{\ell\ell'}^* &= \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{n}\{\psi_\ell(X_{ijk}^{*(\ell)}(\tilde{\boldsymbol{T}}_n^*,O_{p\times J},\tilde{\boldsymbol{s}}_n)) - \bar{\psi}_\ell(X_{\cdots}^{*(\ell)}(\tilde{\boldsymbol{T}}_n^*,O_{p\times J},\tilde{\boldsymbol{s}}_n))\}\\ &\quad\cdot\{\psi_{\ell'}(X_{ijk}^{*(\ell')}(\tilde{\boldsymbol{T}}_n^*,O_{p\times J},\tilde{\boldsymbol{s}}_n)) - \bar{\psi}_{\ell'}(X_{\cdots}^{*(\ell')}(\tilde{\boldsymbol{T}}_n^*,O_{p\times J},\tilde{\boldsymbol{s}}_n))\}/\{IJ(n-1)\},\end{aligned} \tag{2.8}$$

$\tilde{\boldsymbol{T}}_n^* = (\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_1+(\tilde{\boldsymbol{\alpha\beta}})_{11},\cdots,\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_1+(\tilde{\boldsymbol{\alpha\beta}})_{1J},\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_2+(\tilde{\boldsymbol{\alpha\beta}})_{21},\cdots,\tilde{\boldsymbol{\mu}}+\tilde{\boldsymbol{\alpha}}_I+(\tilde{\boldsymbol{\alpha\beta}})_{IJ})$.

$M^{\sharp(\ell)}_{ijj'}(\tilde{\boldsymbol{t}}_{ijj'}, \boldsymbol{\theta}, \tilde{\boldsymbol{s}}_n)$ is nonincreasing in $\theta^{(\ell)}$. Hence for $j \neq j'$, we put

$$\hat{\mu}^{(\ell)}_{ijj'} = [\sup\{\theta^{(\ell)} : M^{\sharp(\ell)}_{ijj'}(\tilde{\boldsymbol{t}}_{ijj'}, \boldsymbol{\theta}, \tilde{\boldsymbol{s}}_n) \geq 0\} + \inf\{\theta^{(\ell)} : M^{\sharp(\ell)}_{ijj'}(\tilde{\boldsymbol{t}}_{ijj'}, \boldsymbol{\theta}, \tilde{\boldsymbol{s}}_n) \leq 0\}]/2, \tag{2.9}$$

where $\tilde{\boldsymbol{t}}_{ijj'} = (\bar{\boldsymbol{X}}_{ij.} + \bar{\boldsymbol{X}}_{ij'.})/2$, $\tilde{\boldsymbol{s}}_n = (\tilde{s}^{(1)}_n, \cdots, \tilde{s}^{(p)}_n)'$. Then, by setting $\boldsymbol{\mu}_{ij} = \boldsymbol{\mu} + \boldsymbol{\alpha}_i + \boldsymbol{\beta}_j + (\boldsymbol{\alpha\beta})_{ij}$, as an robust estimator of $\frac{\boldsymbol{\mu}_{ij} - \boldsymbol{\mu}_{ij'}}{2}$, we propose $\hat{\boldsymbol{\mu}}_{ijj'} = (\hat{\mu}^{(1)}_{ijj'}, \cdots, \hat{\mu}^{(p)}_{ijj'})$. Hence since we get

$$\sum_{j'=1}^{J} (\boldsymbol{\mu}_{ij} - \boldsymbol{\mu}_{ij'})/J = \boldsymbol{\mu}_{ij} - \bar{\boldsymbol{\mu}}_{i.}$$

and

$$\sum_{i=1}^{I} \sum_{j'=1}^{J} (\boldsymbol{\mu}_{ij} - \boldsymbol{\mu}_{ij'})/(IJ) = \bar{\boldsymbol{\mu}}_{.j} - \bar{\boldsymbol{\mu}}_{..},$$

we may propose $(\hat{\boldsymbol{\alpha\beta}})_{ij} = 2\sum_{j'=1}^{J} \hat{\boldsymbol{\mu}}_{ijj'}/J - 2\sum_{i=1}^{I}\sum_{j'=1}^{J} \hat{\boldsymbol{\mu}}_{ijj'}/(IJ)$ and $\hat{\boldsymbol{\beta}}_j = 2\sum_{i=1}^{I}\sum_{j'=1}^{J} \hat{\boldsymbol{\mu}}_{ijj'}/(IJ)$ as robust estimators of $(\boldsymbol{\alpha\beta})_{ij}$ and $\boldsymbol{\beta}_j$, respectively, where we put $\hat{\boldsymbol{\mu}}_{ijj} = 0$.

Lastly since $M^{\natural(\ell)}(\boldsymbol{T}, \boldsymbol{\theta}, \boldsymbol{s})$ is nonincreasing in $\theta^{(\ell)}$, we put

$$\hat{\mu}^{(\ell)} = [\sup\{\theta^{(\ell)} : M^{\natural(\ell)}(\tilde{\boldsymbol{T}}^{\natural}, \boldsymbol{\theta}, \tilde{\boldsymbol{s}}_n) \geq 0\} + \inf\{\theta^{(\ell)} : M^{\natural(\ell)}(\tilde{\boldsymbol{T}}^{\natural}, \boldsymbol{\theta}, \tilde{\boldsymbol{s}}_n) \leq 0\}]/2, \tag{2.10}$$

where $\tilde{\boldsymbol{T}}^{\natural} = (\tilde{\boldsymbol{\alpha}}_1 + \tilde{\boldsymbol{\beta}}_1 + (\tilde{\boldsymbol{\alpha\beta}})_{11}, \cdots, \tilde{\boldsymbol{\alpha}}_1 + \tilde{\boldsymbol{\beta}}_J + (\tilde{\boldsymbol{\alpha\beta}})_{1J}, \tilde{\boldsymbol{\alpha}}_2 + \tilde{\boldsymbol{\beta}}_1 + (\tilde{\boldsymbol{\alpha\beta}})_{21}, \cdots, \tilde{\boldsymbol{\alpha}}_I + \tilde{\boldsymbol{\beta}}_J + (\tilde{\boldsymbol{\alpha\beta}})_{IJ})$. As an M-estimator of $\boldsymbol{\mu}$, we propose $\hat{\boldsymbol{\mu}}_n = (\hat{\mu}^{(1)}, \cdots, \hat{\mu}^{(p)})'$.

3 Robust Statistics and Asymptotic Linearity

We impose the following conditions.

(c.1); $\psi_\ell(x) = \psi_{\ell 1}(x) + \psi_{\ell 2}(x) + \psi_{\ell 3}(x) + \psi_{\ell 4}(x)$, $\psi_{\ell 1}(x)$ and $\psi_{\ell 2}(x)$ are nondecreasing, $\psi_{\ell 3}(x)$ and $\psi_{\ell 4}(x)$ are nonincreasing, $\psi_{\ell 1}(x)$ and $\psi_{\ell 3}(x)$ are continuous, and $\psi_{\ell 2}(x)$ and $\psi_{\ell 4}(x)$ are step functions having only finitely many jumps. There exist constants $b_{\ell i} \leq c_{\ell i}$ such that $\psi_{\ell i}(x) = \psi_{\ell i}(b_{\ell i})$ for $x \leq b_{\ell i}$; $= \psi_{\ell i}(c_{\ell i})$ for $x \geq c_{\ell i}$ $(i = 1, 2, 3, 4)$. □

(c.2); Assume that the ℓ-th marginal density of $F(\boldsymbol{x})$ is defined by $f_\ell(x^{(\ell)})$. Then $f_\ell(x^{(\ell)})'s$ have finite Fisher's informations, i.e., for $\ell = 1, \cdots, p$,

$$0 < \int_{-\infty}^{\infty} \{-f'_\ell(x^{(\ell)})/f_\ell(x^{(\ell)})\}^2 f_\ell(x^{(\ell)}) dx^{(\ell)} < \infty$$

and

$$0 < \int_{-\infty}^{\infty} \{-1 - x^{(\ell)} f'_\ell(x^{(\ell)})/f_\ell(x^{(\ell)})\}^2 f_\ell(x^{(\ell)}) dx^{(\ell)} < \infty. \quad \square$$

(c.3); $F(\boldsymbol{x})$ has finite Fisher's informations, i.e., for $\ell = 1, \cdots, p$,

$$0 < \int_{R^p} \{-\partial f(\boldsymbol{x})/\partial x^{(\ell)}/f(\boldsymbol{x})\}^2 dF(\boldsymbol{x}) < \infty$$

$$0 < \int_{R^p} \{-1 - x^{(\ell)}\partial f(\boldsymbol{x})/\partial x^{(\ell)}/f(\boldsymbol{x})\}^2 dF(\boldsymbol{x}) < \infty. \quad \square$$

If (c.2) is satisfied, the densities $\{\prod_{i=1}^I \prod_{j=1}^J \prod_{k=1}^n [1/(\sigma^{(\ell)} e^{\omega^{(\ell)}/\sqrt{N}}) f_\ell((x_{ijk}^{(\ell)} - \Delta_{ij}^{(\ell)}/\sqrt{N})/(\sigma^{(\ell)} e^{\omega^{(\ell)}/\sqrt{N}}))]\}$ are contiguous to the densities $\{\prod_{i=1}^I \prod_{j=1}^J \prod_{k=1}^n [(1/\sigma^{(\ell)}) f_\ell(x_{ijk}^{(\ell)}/\sigma^{(\ell)})]\}$ as n tends to infinity. Further (c.3) implies that the densities $\{\prod_{i=1}^I \prod_{j=1}^J \prod_{k=1}^n [\{1/\prod_{\ell=1}^p (\sigma^{(\ell)} e^{\omega^{(\ell)}/\sqrt{N}})\} f((x_{ij}^{(1)} - \Delta_{ij}^{(1)}/\sqrt{N})/(\sigma^{(1)} e^{\omega^{(1)}/\sqrt{N}}), \cdots, (x_{ij}^{(p)} - \Delta_{ij}^{(p)}/\sqrt{N})/(\sigma^{(p)} e^{\omega^{(p)}/\sqrt{N}}))]\}$ are contiguous to the densities $\{\prod_{i=1}^I \prod_{j=1}^J \prod_{k=1}^n [(1/\prod_{\ell=1}^p \sigma^{(\ell)}) f(x_{ij}^{(1)}/\sigma^{(1)}, \cdots, x_{ij}^{(p)}/\sigma^{(p)})]\}$.

We can derive asymptotic linearity for the statistics $M_{ij}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})$, $M_j^{*(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}^*, \boldsymbol{s})$, $M_{ijj'}^{\sharp(\ell)}(\boldsymbol{t}, \boldsymbol{\theta}, \boldsymbol{s})$ and $M^{\natural(\ell)}(\boldsymbol{T}, \boldsymbol{\theta}, \boldsymbol{s})$. For $p \times (IJ)$ matrices $\boldsymbol{T}$, $\boldsymbol{\Theta}$ and p dimensional column vector $\boldsymbol{s}$, we introduce the statistic

$$T_{ij}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) = \sum_{k=1}^n \{\psi_\ell(X_{ijk}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})) - \bar{\psi}_\ell\}/\sqrt{n},$$

where

$$\bar{\psi}_\ell = \textstyle\int_{-\infty}^\infty \psi_\ell(\frac{x}{\rho^{(\ell)}}) dF_\ell(\frac{x}{\sigma^{(\ell)}}). \tag{3.1}$$

Proceeding as in the proof of Lemma3.1 of Shiraishi (1996), we get Lemma 3.1.

Lemma 3.1. Let $(\boldsymbol{X}_{111}, \cdots, \boldsymbol{X}_{IJn})$ have joint distribution function $\prod_{i=1}^I \prod_{j=1}^J \prod_{k=1}^n F(x_{ijk}^{(1)}/\sigma^{(1)}, \cdots, x_{ijk}^{(p)}/\sigma^{(p)})$. Then under the conditions (c.1)-(c.3), for any positive ϵ, C_1, C_2 and C_3,

$$\lim_{n\to\infty} P\{ \sup_{\|\boldsymbol{\eta}^{(\ell)}\|_{IJ} < C_1, \|\boldsymbol{\Delta}^{(\ell)}\|_{IJ} < C_2, |\omega^{(\ell)}| < C_3} |T_{ij}^{(\ell)}(\boldsymbol{\eta}/\sqrt{n}, \boldsymbol{\Delta}/\sqrt{n}, g(\boldsymbol{\rho}, \boldsymbol{\omega}/\sqrt{n})) - T_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho})$$
$$+ d_\ell(\eta_{ij}^{(\ell)} + \Delta_{ij}^{(\ell)})/\sigma^{(\ell)} + \omega^{(\ell)} e_\ell| > \epsilon\} = 0,$$

where $\boldsymbol{\eta} = (\boldsymbol{\eta}_{11}, \cdots, \boldsymbol{\eta}_{1J}, \boldsymbol{\eta}_{21}, \cdots, \boldsymbol{\eta}_{IJ})$, $\boldsymbol{\eta}_{ij} = (\eta_{ij}^{(1)}, \cdots, \eta_{ij}^{(p)})'$, $\boldsymbol{\eta}^{(\ell)} = (\eta_{11}^{(\ell)}, \cdots, \eta_{1J}^{(\ell)}, \eta_{21}^{(\ell)}, \cdots, \eta_{IJ}^{(\ell)})$, $\boldsymbol{\Delta}$ is also the definition similar to $\boldsymbol{\eta}$, $\boldsymbol{\rho} = (\rho^{(1)}, \cdots, \rho^{(p)})'$, $g(\boldsymbol{\rho}, \boldsymbol{\omega}/\sqrt{n}) = (\rho^{(1)} e^{\omega^{(1)}/\sqrt{n}}, \cdots, \rho^{(p)} e^{\omega^{(p)}/\sqrt{n}})'$, $\boldsymbol{\omega} = (\omega^{(1)}, \cdots, \omega^{(p)})'$, $\|\boldsymbol{z}\|_m = \sqrt{\boldsymbol{z} \cdot \boldsymbol{z}'}$ for the m-dimensional row vector $\boldsymbol{z}$, $d_\ell = -\int_{-\infty}^\infty \psi_\ell(\sigma^{(\ell)} x/\rho^{(\ell)}) f_\ell'(x) dx$, and $e_\ell = -\int_{-\infty}^\infty \psi_\ell(\sigma^{(\ell)} x/\rho^{(\ell)}) \{1 + \frac{x f_\ell'(x)}{f_\ell(x)}\} f_\ell(x) dx$. $\square$

By using $T_{ij}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})$, we can rewrite

$$M_{ij}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) = T_{ij}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) - \bar{T}_{i\cdot}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) - \bar{T}_{\cdot j}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}) + \bar{T}_{\cdot\cdot}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s}), \tag{3.2}$$

where $\bar{T}_{i\cdot}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})$, $\bar{T}_{\cdot j}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})$ and $\bar{T}_{\cdot\cdot}^{(\ell)}(\boldsymbol{T}, \boldsymbol{\Theta}, \boldsymbol{s})$ are respectively sample means about the dot $\cdot$.

Theorem 3.2. Let $B_1(C) = \{\boldsymbol{\Delta}; \sum_{i=1}^I \boldsymbol{\Delta}_{ij} = \sum_{j=1}^J \boldsymbol{\Delta}_{ij} = \boldsymbol{0}$ for all $i, j's$, $\|\boldsymbol{\Delta}^{(\ell)}\|_{IJ} < C\}$ and $B_2(C) =$

$\boldsymbol{\Delta}$. Then under the assumptions of Lemma 3.1, for any positive ϵ and C,

$$\lim_{n\to\infty} P\{\sup_{\boldsymbol{\Delta}\in B_1(C)} |M_{ij}^{(\ell)}(\tilde{\boldsymbol{T}}_n, \boldsymbol{\Delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) + d_\ell \cdot \Delta_{ij}^{(\ell)}/\sigma^{(\ell)}| > \epsilon\} = 0,$$

where $\boldsymbol{\rho} = \sqrt{\frac{\pi}{2}} \cdot (\int_{-\infty}^{\infty} |x| dF_1(\frac{x}{\sigma^{(1)}}), \cdots, \int_{-\infty}^{\infty} |x| dF_p(\frac{x}{\sigma^{(p)}}))'$.

Proof. From the asymptotic boundedness of $n \cdot \sum_{i=1}^I \sum_{j=1}^J (\tilde{\mu}^{(\ell)} + \tilde{\alpha}_i^{(\ell)} + \tilde{\beta}_j^{(\ell)})^2$, for any $\epsilon_0 > 0$, we can take K_1 and K_2 sufficiently large such that $P\{n \sum_{i=1}^I \sum_{j=1}^J (\tilde{\mu}^{(\ell)} + \tilde{\alpha}_i^{(\ell)} + \tilde{\beta}_j^{(\ell)})^2 \geq K_1^2\} < \frac{\epsilon_0}{2}$ and $P\{\sqrt{n} | \log(\tilde{s}_n^{(\ell)}) - \log(\rho^{(\ell)})| \geq K_2\} < \frac{\epsilon_0}{2}$. Hence we get

$P\{\sup_{\boldsymbol{\Delta}\in B_1(C)} |M_{ij}^{(\ell)}(\tilde{\boldsymbol{T}}_n, \boldsymbol{\Delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) + d_\ell \cdot \Delta_{ij}^{(\ell)}| > \epsilon\}$
$\leq P\{\sup_{\boldsymbol{\Delta}\in B_1(C)} |M_{ij}^{(\ell)}(\tilde{\boldsymbol{T}}_n, \boldsymbol{\Delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) + d_\ell \cdot \Delta_{ij}^{(\ell)}/\sigma^{(\ell)}| > \epsilon,$
$n \sum_{i=1}^I \sum_{j=1}^J (\bar{X}_{i..}^{(\ell)} + \bar{X}_{.j.}^{(\ell)} - \bar{X}_{...}^{(\ell)})^2 < K_1^2,\ \sqrt{n} | \log(\tilde{s}_n^{(\ell)}) - \log(\rho^{(\ell)})| < K_2\}$
$+P\{n \sum_{i=1}^I \sum_{j=1}^J (\bar{X}_{i..}^{(\ell)} + \bar{X}_{.j.}^{(\ell)} - \bar{X}_{...}^{(\ell)})^2 \geq K^2\} + P\{\sqrt{n} | \log(\tilde{s}_n^{(\ell)}) - \log(\rho^{(\ell)})| \geq K_2\}$
$\leq P\{\sup_{\boldsymbol{\eta}\in B_2(K_1), \boldsymbol{\Delta}\in B_1(C), |\omega^{(\ell)}|<K_2} |M_{ij}^{(\ell)}(\boldsymbol{\eta}/\sqrt{n}, \boldsymbol{\Delta}/\sqrt{n}, g(\boldsymbol{\rho}, \boldsymbol{\omega}/\sqrt{n})) - M_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) + d_\ell \cdot$
$\Delta_{ij}^{(\ell)}/\sigma^{(\ell)}| > \epsilon\} + \epsilon_0$.

Therefore Lemma 3.1 shows the conclusion. □

We get Corollary 3.3 as a direct result of Theorem 3.2.

Corollary 3.3. Under the assumptions of Lemma 3.1, for any positive ϵ and C_1,

$$\lim_{n\to\infty} P\{\sup_{\boldsymbol{\Delta}\in B(C)} ||M_{ij}^{(\ell)}(\tilde{\boldsymbol{T}}_n, \boldsymbol{\Delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n)| - |M_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) - d_\ell \cdot \Delta_{ij}^{(\ell)}/\sigma^{(\ell)}|| > \epsilon\} = 0. \quad \square$$

Next by using Lemma corresponding to Lemma 3.1, proceeding as in the proof of Theorem 3.2, we get Theorems 3.4-3.6.

Theorem 3.4. Let $B^*(C) = \{\boldsymbol{\Delta}^* = (\boldsymbol{\Delta}_1^*, \cdots, \boldsymbol{\Delta}_J^*); \sum_{j=1}^J \boldsymbol{\Delta}_j^* = \mathbf{0},\ \|\boldsymbol{\Delta}^{*(\ell)}\|_J < C\}$. Then under the assumptions of Lemma 3.1, for any positive ϵ and C,

$$\lim_{n\to\infty} P\{\sup_{\boldsymbol{\Delta}^*\in B^*(C)} |M_j^{*(\ell)}(\tilde{\boldsymbol{T}}_n^*, \boldsymbol{\Delta}^*/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M_j^{*(\ell)}(O_{p\times IJ}, O_{p\times J}, \boldsymbol{\rho}) + I d_\ell \cdot \Delta_j^{*(\ell)}/\sigma^{(\ell)}| > \epsilon\} = 0,$$

where $\tilde{\boldsymbol{T}}_n^*$ is defined in (2.8). □

Theorem 3.5. Under the assumptions of Lemma 3.1, for any positive ϵ and C,

$$\lim_{n\to\infty} P\{\sup_{\|\boldsymbol{\delta}\|_p<C} |M_{ijj'}^{\sharp(\ell)}(\tilde{\boldsymbol{t}}_{ijj'}, \boldsymbol{\delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M_{ijj'}^{\sharp(\ell)}(O_{p\times 1}, O_{p\times 1}, \boldsymbol{\rho}) + 2d_\ell \cdot \delta^{(\ell)}/\sigma^{(\ell)}| > \epsilon\} = 0,$$

Theorem 3.6. Assume that $f_\ell(-x) = f_\ell(x)$ and $\psi_\ell(-x) = -\psi_\ell(x)$ for all ℓ's. Then under the assumptions of Lemma 3.1, for any positive ϵ and C,

$$\lim_{n\to\infty} P\{\sup_{\|\boldsymbol{\delta}\|_p<C} |M^{\natural(\ell)}(\tilde{\boldsymbol{T}}_n^\natural, \boldsymbol{\delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M^{\natural(\ell)}(O_{p\times IJ}, O_{p\times 1}, \boldsymbol{\rho}) + IJd_\ell \cdot \delta^{(\ell)}/\sigma^{(\ell)}| > \epsilon\} = 0,$$

where $\tilde{\boldsymbol{T}}_n^\natural$ is defined in (2.10). □

4 Asymptotic Properties of Test Statistics

4.1 Interactions

Since the distributions of statistics under the model (1.1) do not depend on $\boldsymbol{\mu}$, $\boldsymbol{\alpha}_i$'s, and $\boldsymbol{\beta}_j$'s, throughout this Section, it is assumed without any loss of generality that

$$\boldsymbol{\mu} = \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_I = \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_J = \mathbf{0}. \tag{4.1}$$

Based on the asymptotic distribution of $\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, O_{p\times IJ}, \tilde{\boldsymbol{s}}_n)$ under the null hypothesis H, we consider to test H versus the alternative A.

Lemma 4.1. Suppose that conditions (c.1)-(c.3) are satisfied. Then under H, as $n \to \infty$, $\mathrm{vec}(\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, O_{p\times IJ}, \tilde{\boldsymbol{s}}_n))$ has asymptotically a pIJ-variate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\Lambda \otimes \Gamma$, where $\Lambda = (\lambda_{mm'})_{m,m'=1,\cdots,IJ}$, $\lambda_{mm'} = (1-1/I)(1-1/J)$ if $m = m'$; $= -1/J + 1/(IJ)$ if $m = (i-1)J + j$ and $m' = (i-1)J + j'$ for i and (j, j') such that $1 \le i \le I$ and $1 \le j \ne j' \le J$; $= -1/I + 1/(IJ)$ if $m = (i-1)J + j$ and $m' = (i'-1)J + j$ for (i, i') and j such that $1 \le i \ne i' \le I$ and $1 \le j \le J$; $= 1/(IJ)$ elsewhere,

$$\Gamma = (\gamma_{\ell\ell'})_{\ell,\ell'=1,\cdots,p}, \tag{4.2}$$

$\gamma_{\ell\ell'} = \int_{-\infty}^{\infty} \{\psi_\ell(\frac{x}{\rho^{(\ell)}}) - \bar{\psi}_\ell\}^2 dF_\ell(\frac{x}{\sigma^{(\ell)}})$ if $\ell = \ell'$; $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\psi_\ell(\frac{x}{\rho^{(\ell)}}) - \bar{\psi}_\ell\}\{\psi'_\ell(\frac{y}{\rho^{(\ell')}}) - \bar{\psi}'_\ell\} dF_{\ell\ell'}(\frac{x}{\sigma^{(\ell)}}, \frac{y}{\sigma^{(\ell')}})$ elsewhere, $\bar{\psi}_\ell$ is defined by (3.1), $F_{\ell\ell'}(x, y)$ stands for the $(\ell, \ell')-$th marginal distribution of $F(\boldsymbol{x})$ and $\otimes$ denotes Kronecker's product.

Proof. Theorem 3.2 shows that $\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, O_{p\times IJ}, \tilde{\boldsymbol{s}}_n) - \boldsymbol{M}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) \xrightarrow{\mathrm{P}} \mathbf{0}$ under H. Using Cramer-Wold technique, it follows that $\mathrm{vec}(\boldsymbol{M}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) \xrightarrow{\mathrm{L}} N_{pIJ}(\mathbf{0}, \Lambda\otimes\Gamma)$, where $\xrightarrow{\mathrm{L}} N_m(\boldsymbol{\mu}, \Sigma)$ denotes convergence in law to an m variate normal distribution with mean $\boldsymbol{\mu}$ and variance-covariance matrix Σ. Hence the conclusion is found. □

$$(4.3)\qquad A_n;\ (\boldsymbol{\alpha\beta})_{ij} = \boldsymbol{\Delta}_{ij}/\sqrt{n},\ \boldsymbol{\Delta}_{ij} \neq \boldsymbol{\Delta}_{i'j'}\ \textit{for some}\ (i,j) \neq (i',j'),$$

where $\sum_{i=1}^{I} \boldsymbol{\Delta}_{ij} = \sum_{j=1}^{J} \boldsymbol{\Delta}_{ij} = \mathbf{0}$ for all i, j's and $\boldsymbol{\Delta}_{ij} = (\Delta_{ij}^{(1)}, \cdots, \Delta_{ij}^{(p)})'$. When $\boldsymbol{\Delta}_{ij} = \mathbf{0}$ for all i, j's, A_n is equivalent to the null hypothesis H.

Theorem 4.2. Suppose that conditions (c.1)-(c.3) are satisfied and Γ is positive definite. Then under A_n, as $n \to \infty$, MT has asymptotically a noncentral χ^2-distribution with $p(I-1)(J-1)$ degrees of freedom and noncentrality parameter δ^2, where
$\delta^2 = \sum_{i=1}^{I} \sum_{j=1}^{J} \boldsymbol{\mu}_{ij}'\Gamma^{-1}\boldsymbol{\mu}_{ij}$, $\boldsymbol{\mu}_{ij} = (\mu_{ij}^{(1)}, \cdots, \mu_{ij}^{(p)})'$ and $\mu_{ij}^{(\ell)} = d_\ell \Delta_{ij}^{(\ell)}/\sigma^{(\ell)}$.

Proof. From Theorem 3.2, we get under H

$$(4.4)\qquad |M_{ij}^{(\ell)}(\tilde{\boldsymbol{T}}_n, -\boldsymbol{\Delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n) - M_{ij}^{(\ell)}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}) - \mu_{ij}^{(\ell)}| \xrightarrow{\text{P}} 0.$$

Here it follows that $\text{vec}\{\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, -\boldsymbol{\Delta}/\sqrt{n}, \tilde{\boldsymbol{s}}_n)\} \xrightarrow{\text{L}} N(\text{vec}(\boldsymbol{\mu}), \Lambda \otimes \Gamma)$ under H, which implies the relation that

$$(4.5)\qquad \text{vec}(\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, O_{p\times IJ}, \tilde{\boldsymbol{s}}_n)) \xrightarrow{\text{L}} N_{pIJ}(\text{vec}(\boldsymbol{\mu}), \Lambda \otimes \Gamma)\ \ \textit{under} A_n,$$

where $\boldsymbol{\mu} = (\boldsymbol{\mu}_{11}, \cdots, \boldsymbol{\mu}_{IJ})$. Condition (c.1) shows $\hat{\Gamma} \xrightarrow{\text{P}} \Gamma$ under A_n. Combining this with (4.5), from Theorem stated in Section 3.5 of Serfling (1980), we find that $MT = \text{vec}(\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, O_{p\times IJ}, \tilde{\boldsymbol{s}}_n))'(\Lambda^- \otimes \hat{\Gamma}^{-1})\text{vec}(\boldsymbol{M}(\tilde{\boldsymbol{T}}_n, O_{p\times IJ}, \tilde{\boldsymbol{s}}_n))$ has asymptotically a noncentral χ^2-distribution. Furthermore the degree of freedom is equal to the rank of $\Lambda \otimes \Gamma$ which is $p(I-1)(J-1)$ and the noncentrality parameter is given by $\text{vec}(\boldsymbol{\mu})'(\Lambda \otimes \Gamma)^- \text{vec}(\boldsymbol{\mu}) = \text{vec}(\boldsymbol{\mu})'(E_{IJ} \otimes \Gamma^{-1})\text{vec}(\boldsymbol{\mu}) = \delta^2$. Therefore the conclusion is found. □

4.2 Main Effects

Based on the asymptotic distribution of $\boldsymbol{M}^*(\tilde{\boldsymbol{T}}_n^*, O_{p\times J}, \tilde{\boldsymbol{s}}_n)$ under H^*, we consider to test the null hypothesis H^* versus the alternative A^*. By using Theorem 3.4, proceeding as in the proof of Lemma 4.1, we get

Lemma 4.3. Suppose that conditions (c.1)-(c.3) are satisfied. Then under H^*, as $n \to \infty$,
$\text{vec}(\boldsymbol{M}^*(\tilde{\boldsymbol{T}}_n^*, O_{p\times J}, \tilde{\boldsymbol{s}}_n))$ has asymptotically a pJ-variate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $I \cdot \Lambda_J^* \otimes \Gamma$, where

$$(4.6)\qquad \Lambda_J^* = E_J - \mathbf{1}_J \cdot \mathbf{1}_J'/J,$$

We consider the sequence of local alternatives

$$A_n^*;\ \boldsymbol{\beta}_j = \boldsymbol{\Delta}_j^*/\sqrt{n},\ \ \boldsymbol{\Delta}_j^* \neq \boldsymbol{\Delta}_{j'}^*\ \textit{for some}\ j \neq j', \tag{4.7}$$

where $\sum_{j=1}^{J} \boldsymbol{\Delta}_j^* = \mathbf{0}$.

Using Lemma 4.3 and Theorem 3.4, we get Theorem 4.4 similar to Theorem 4.2.

Theorem 4.4. Suppose that the conditions of Lemma 4.3 are satisfied. Then under A_n^*, as $n \to \infty$, MT^* has asymptotically a noncentral χ^2-distribution with $p(J-1)$ degrees of freedom and noncentrality parameter δ^{*2}, where $\delta^{*2} = I\sum_{j=1}^{J}(\boldsymbol{\nu}_j^*)'\Gamma^{-1}\boldsymbol{\nu}_j^*$, $\boldsymbol{\nu}_j^* = (\nu_j^{*(1)}, \cdots, \nu_j^{*(p)})'$ and $\nu_j^{*(\ell)} = d_\ell \Delta_j^{*(\ell)}/\sigma^{(\ell)}$. □

5 Properties for Point Estimates

We proposed $(\hat{\boldsymbol{\alpha\beta}}) = ((\hat{\boldsymbol{\alpha\beta}})_{11}, \cdots, (\hat{\boldsymbol{\alpha\beta}})_{1J}, (\hat{\boldsymbol{\alpha\beta}})_{21}, \cdots, (\hat{\boldsymbol{\alpha\beta}})_{IJ})$ and $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_1, \cdots, \hat{\boldsymbol{\beta}}_J)$ as estimators of matrix $(\boldsymbol{\alpha\beta})$ and $\boldsymbol{\beta}$, respectively. We add the condition (c.1') and (c.4).

(c.1'); $\psi_\ell(x) = \psi_{\ell 1}(x) + \psi_{\ell 2}(x)$, $\psi_{\ell 1}(x)$ and $\psi_{\ell 2}(x)$ are nondecreasing, $\psi_{\ell 1}(x)$ is continuous, and $\psi_{\ell 2}(x)$ is step functions having only finitely many jumps. There exist constants $b_{\ell i} \leq c_{\ell i}$ such that $\psi_{\ell i}(x) = \psi_{\ell i}(b_{\ell i})$ for $x \leq b_{\ell i}$; $= \psi_{\ell i}(c_{\ell i})$ for $x \geq c_{\ell i}$ $(i = 1, 2, 3, 4)$. □

(c.4); $d_\ell > 0$ for $\ell = 1, \cdots, p$. □

Then we get

Theorem 5.1. Suppose that the conditions (c.1') and (c.2)-(c.4) are satisfied. Then $\sqrt{n} \cdot \mathrm{vec}((\hat{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))$ has asymptotically a pIJ-variate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\Lambda \otimes (D\Gamma D)$, where $D = \mathrm{diag}(\sigma^{(1)}/d_1, \cdots, \sigma^{(p)}/d_p)$.

Proof. If we put $(\hat{\boldsymbol{\alpha\beta}}) = (\hat{\boldsymbol{\alpha\beta}})(\boldsymbol{X}_{111}, \cdots, \boldsymbol{X}_{IJn})$, we may show

$$(\hat{\boldsymbol{\alpha\beta}})(\boldsymbol{X}_{111}, \cdots, \boldsymbol{X}_{IJn}) = (\hat{\boldsymbol{\alpha\beta}})(\boldsymbol{e}_{111}, \cdots, \boldsymbol{e}_{IJn}) + (\boldsymbol{\alpha\beta}),$$

where $(\boldsymbol{e}_{111}, \cdots, \boldsymbol{e}_{IJn})$ is defined in (1.1). We assume for convenience

$$\boldsymbol{\mu} = \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_I = \boldsymbol{\beta}_1 = \cdots = \boldsymbol{\beta}_J = \boldsymbol{\alpha\beta}_{11} = \cdots = \boldsymbol{\alpha\beta}_{IJ} = \mathbf{0}. \tag{5.1}$$

Let us define the solution of system of the following equations by $\hat{\mu}_{ijj'}^{*(\ell)}$.

$$M_{ijj'}^{\sharp(\ell)}(O_{p\times 1}, O_{p\times 1}, \boldsymbol{\rho}) = 2\sqrt{n}d_\ell \cdot \mu_{ijj'}^{(\ell)}/\sigma^{(\ell)} \text{ for all } \ell\text{'s, } i\text{'s, } j\text{'s and } j'\text{'s.} \tag{5.2}$$

$$\sqrt{n}|\hat{\mu}^{(\ell)}_{ijj'} - \hat{\mu}^{*(\ell)}_{ijj'}| \xrightarrow{\text{P}} 0 \text{ for all } i\text{'s}, j\text{'s and } j'\text{'s}. \tag{5.3}$$

By combining the definition of $\hat{(\boldsymbol{\alpha}\boldsymbol{\beta})}_{ij}$ with (5.3), it follows that

$$\sqrt{n}\hat{(\boldsymbol{\alpha}\boldsymbol{\beta})}_{ij} \sim \sqrt{n}\{2\sum_{j'=1}^{J} \hat{\boldsymbol{\mu}}^*_{ijj'}/J - 2\sum_{i=1}^{I}\sum_{j'=1}^{J} \hat{\boldsymbol{\mu}}^*_{ijj'}/(IJ)\}, \tag{5.4}$$

where $\hat{\boldsymbol{\mu}}^*_{ijj'} = (\mu^{*(1)}_{ijj'}, \cdots, \hat{\mu}^{*(p)}_{ijj'})'$ and $V_N \sim W_N$ denotes $V_N - W_N \xrightarrow{\text{P}} 0$. By using (5.2) and (5.4),

$$\sqrt{n}\hat{(\boldsymbol{\alpha}\boldsymbol{\beta})} \sim D\boldsymbol{M}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho}), \tag{5.5}$$

$\boldsymbol{M}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho})$ is defined by (2.5). Since, by using the multivariate central limit theorem, $\text{vec}\{D\boldsymbol{M}(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho})\}$ has asymptotically $N_{pIJ}(\mathbf{0}, \Lambda \otimes (D\Gamma D))$, (5.5) implies $\sqrt{n}\cdot\text{vec}\{\hat{(\boldsymbol{\alpha}\boldsymbol{\beta})}\} \xrightarrow{\text{L}} N_{pIJ}(\mathbf{0}, \Lambda \otimes (D\Gamma D))$. Hence the assertion is established. □

Theorem 5.2. Suppose that the conditions (c.1') and (c.2)-(c.4) are satisfied. Then $\sqrt{n}\cdot\text{vec}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ has asymptotically a pJ-variate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $\Lambda^*_J \otimes (D\Gamma D)$.

Proof. If we put $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}(\boldsymbol{X}_{111}, \cdots, \boldsymbol{X}_{IJn})$, we may show

$$\hat{\boldsymbol{\beta}}(\boldsymbol{X}_{111}, \cdots, \boldsymbol{X}_{IJn}) = \hat{\boldsymbol{\beta}}(\boldsymbol{e}_{111}, \cdots, \boldsymbol{e}_{IJn}) + \boldsymbol{\beta}.$$

We assume (5.1) for convenience. By combining the definition of $\hat{\boldsymbol{\beta}}_j$ with (5.3), it follows that

$$\sqrt{n}\hat{\boldsymbol{\beta}}_j \sim 2\sqrt{n}\sum_{i=1}^{I}\sum_{j'=1}^{J} \hat{\boldsymbol{\mu}}^*_{ijj'}/(IJ), \tag{5.6}$$

where $\hat{\boldsymbol{\mu}}^*_{ijj'} = (\mu^{*(1)}_{ijj'}, \cdots, \hat{\mu}^{*(p)}_{ijj'})'$ is defined in (5.2). By using (5.2) and (5.6),

$$\sqrt{n}\hat{\boldsymbol{\beta}} \sim D\boldsymbol{M}^*(O_{p\times IJ}, O_{p\times J}, \boldsymbol{\rho})/I, \tag{5.7}$$

$\boldsymbol{M}^*(O_{p\times IJ}, O_{p\times J}, \boldsymbol{\rho})$ is defined by (2.7). Since, by using the multivariate central limit theorem, $\text{vec}\{D\boldsymbol{M}^*(O_{p\times IJ}, O_{p\times IJ}, \boldsymbol{\rho})/I\}$ has asymptotically $N_{pJ}(\mathbf{0}, \Lambda^*_J \otimes (D\Gamma D)/I)$, (5.7) implies $\sqrt{n}\cdot\text{vec}\{\hat{\boldsymbol{\beta}}\} \xrightarrow{\text{L}} N_{Jp}(\mathbf{0}, \Lambda^*_J \otimes (D\Gamma D)/I)$. Hence the assertion is established. □

Theorem 5.3. Assume that $\psi^*_\ell(-x) = -\psi^*_\ell(x)$ for all ℓ's and $f(\boldsymbol{x})$ is diagonally symmetric. Then under the conditions (c.1') and (c.2)-(c.4), $\sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})$ has asymptotically a p-variate normal distribution with mean $\mathbf{0}$ and variance-covariance matrix $D\Gamma D/(IJ)$. Furthermore $\sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})$, $\sqrt{n}\cdot\text{vec}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})$, $\sqrt{n}\cdot\text{vec}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})$ and $\sqrt{n}\cdot\text{vec}(\hat{(\boldsymbol{\alpha}\boldsymbol{\beta})} - (\boldsymbol{\alpha}\boldsymbol{\beta}))$ are asymptotically independent.

Proof. We assume (5.1) for convenience. By using Theorem 3.6, as in the proof of Theorem 5.2, we can show

$$\sqrt{n}\hat{\boldsymbol{\mu}} \sim \frac{1}{IJ} D(M^{\natural(1)}(O_{p\times IJ}, O_{p\times 1}, \boldsymbol{\rho}), \cdots, M^{\natural(p)}(O_{p\times IJ}, O_{p\times 1}, \boldsymbol{\rho})). \tag{5.8}$$

Since the multivariate central limit theorem implies that $D(M^{\natural(1)}(O_{p\times IJ}, O_{p\times 1}, \boldsymbol{\rho}), \cdots, M^{\natural(p)}(O_{p\times IJ}, O_{p\times 1}, \boldsymbol{\rho}))$ has asymptotically $N_p(\mathbf{0}, IJ \cdot D\Gamma D)$, (5.8) implies $\sqrt{n} \cdot \hat{\boldsymbol{\mu}} \xrightarrow{\text{L}} N_p(\mathbf{0}, D\Gamma D/(IJ))$. Hence the first assertion is established. The asymptotic covariances of $\sqrt{n} \cdot \hat{\boldsymbol{\mu}}$, $\sqrt{n} \cdot \text{vec}\{\hat{\boldsymbol{\alpha}}\}$, $\sqrt{n} \cdot \text{vec}\{\hat{\boldsymbol{\beta}}\}$ and $\sqrt{n} \cdot \text{vec}\{\hat{\boldsymbol{\alpha\beta}}\}$ are null. Therefore the second assertion is found. □

6 Asymptotic Robustness

We investigate the asymptotic relative efficiencies (ARE's) of the proposed tests and estimators with respect to the normal theory tests and the LSE's. Since the ARE's for $p \geq 2$ are complicated, at first we give the ARE's for $p = 1$. The normal theory F-tests were reviewed in Chapter 7 of Dunn and Clark [1]. It is simple to verify that (the normalized likelihood ratio F-test statistic) $\xrightarrow{\text{L}} \chi^2_{(I-1)(J-1)}$ under H and $\xrightarrow{\text{L}} \chi^2_{(I-1)(J-1)}(\eta^2)$ under A_n, where $\eta^2 = \boldsymbol{\Delta} \cdot \boldsymbol{\Delta}'/\sigma^2$. Also we can find that $\sqrt{n}(\bar{X}_{11\cdot} - \bar{X}_{1\cdot\cdot} - \bar{X}_{\cdot 1\cdot} + \bar{X}_{\cdots}, \bar{X}_{12\cdot} - \bar{X}_{1\cdot\cdot} - \bar{X}_{\cdot 2\cdot} + \bar{X}_{\cdots}, \cdots, \bar{X}_{IJ\cdot} - \bar{X}_{I\cdot\cdot} - \bar{X}_{\cdot J\cdot} + \bar{X}_{\cdots})'$ $\xrightarrow{\text{L}} N_{IJ}(\mathbf{0}, \sigma^2 \cdot \Lambda)$. Combining these facts with Theorems 4.2 and 5.1, we get

$$\begin{aligned} ARE(MT, F-test) &= ARE((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}})) \\ &= \{\int_{-\infty}^{\infty} \psi_1(\frac{x}{\rho}) f'(\frac{x}{\sigma}) dx\}^2 / \int_{-\infty}^{\infty} \{\psi_1(\frac{x}{\rho}) - \bar{\psi}_1\}^2 dF(\frac{x}{\sigma}), \end{aligned} \tag{6.1}$$

where $ARE(C, D)$ stands for the asymptotic relative efficiency of C with respect to D. Assume that $\psi_1(-x) = -\psi_1(x)$ and $f(-x) = f(x)$ for all x. Then the ARE is equivalent to the classical ARE-result of the one-sample M-estimator with respect to the one-sample mean.

Furthermore when the class of $\psi_1(x)$ is restricted to $\{\psi_1(x) : \psi_1(-x) = -\psi_1(x),\ \psi_1(x) = \psi_1(d) \text{ for } x > d > 0\}$, the choice of $\psi_1(x)$ which gives maximin asymptotic power of the M-test over the class of the distributions that $f(x)$ is in ϵ-contamination neighborhood $\{f(x) = (1-\epsilon)f_0(x) + \epsilon h(x)$: $f_0(x)$ is a fixed symmetric density and $h(x)$ is any symmetric density $\}$ and minimax asymptotic variance of the M-estimator is reviewed by Section 4.6 of Huber (1981). Also the choice of $\psi_1(x)$ over the class of the distributions that $f(x)$ is in ϵ-Kolmogorov neighborhood $\{f(x) : \sup_x |F(x) - F_0(x)| \leq \epsilon$, $F_0(x)$ is a fixed distribution function, $f(-x) = f(x) = F'(x)$, and tests and estimators for main effects remains the same too.

Next we investigate ARE's of the proposed estimators with respect to the LSE's for $p \geq 2$. For two sequences of estimators $\{\boldsymbol{U}_n\}$ and $\{\boldsymbol{V}_n\}$ for $\boldsymbol{\theta}$ so that $\sqrt{n}(\boldsymbol{U}_n - \boldsymbol{\theta})$ and $\sqrt{n}(\boldsymbol{V}_n - \boldsymbol{\theta})$ have asymptotically mp-variate normal distributions with null mean vector and variance-covariance matrix $\boldsymbol{\Sigma}_0 \otimes \boldsymbol{\Sigma}_U$ and $\boldsymbol{\Sigma}_0 \otimes \boldsymbol{\Sigma}_V$ respectively, Shiraishi (1989) defined the ARE of $\boldsymbol{U}_n$ relative to $\boldsymbol{V}_n$ by $\mathrm{ARE}(\boldsymbol{U}_n, \boldsymbol{V}_n) = \{|\boldsymbol{\Sigma}_V|/|\boldsymbol{\Sigma}_U|\}^{1/p}$, where $\boldsymbol{\Sigma}_U$ and $\boldsymbol{\Sigma}_V$ are nonsingular $p \times p$ matrices. In the case p=1, this definition of the ARE is equivalent to the Pitman ARE. When $\boldsymbol{\Sigma}_0$ is nonsingular, it is equal to the ARE defined by Puri and Sen (1985). After simple argument, we can draw $\sqrt{n} \cdot \mathrm{vec}((\tilde{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta})) \xrightarrow{\mathrm{L}} N(\boldsymbol{0}, \Lambda \otimes \mathrm{Var}(\boldsymbol{e}_{111}))$, where $\mathrm{Var}(\boldsymbol{e}_{111})$ stands for the variance-covariance matrix. Combining this fact with Theorem 5.1, we get $\mathrm{ARE}((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}})) = \{|\mathrm{Var}(\boldsymbol{e}_{111})|/|(D\Gamma D)|\}^{1/p}$. Especially when $F(\boldsymbol{x}) = \prod_{\ell=1}^{p} F_\ell(x^{(\ell)})$ for $\boldsymbol{x} = (x^{(1)}, \cdots, x^{(p)})$, the ARE is given by

$$\mathrm{ARE}((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}})) = \mathrm{ARE}(\hat{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}}) = \left(\prod_{\ell=1}^{p} [\{\int_{-\infty}^{\infty} \psi_\ell(\frac{x}{\rho^{(\ell)}}) f'_\ell(\frac{x}{\sigma^{(\ell)}}) dx\}^2 / \int_{-\infty}^{\infty} \{\psi_\ell(\frac{x}{\rho^{(\ell)}}) - \bar{\psi}_\ell\}^2 dF_\ell(\frac{x}{\sigma^{(\ell)}})] \right)^{\frac{1}{p}},$$

which is nearly equal to (6.1).

7 Simulation Study for Estimators

The risk for an estimator $(\hat{\boldsymbol{\alpha\beta}})$ of $(\boldsymbol{\alpha\beta})$ is defined by $E\{\mathrm{vec}((\hat{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))'\mathrm{vec}((\hat{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))\}$. Hence we define the relative risk efficiency of $(\hat{\boldsymbol{\alpha\beta}})$ with respect to $(\tilde{\boldsymbol{\alpha\beta}})$ by $E\{\mathrm{vec}((\tilde{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))'\mathrm{vec}((\tilde{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))\}/E\{\mathrm{vec}((\hat{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))'\mathrm{vec}((\hat{\boldsymbol{\alpha\beta}}) - (\boldsymbol{\alpha\beta}))\}$, which is denoted by $RRE((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}}))$. Under a suitable condition, we may find

$$\lim_{n \to \infty} RRE((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}})) = ARE((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}})).$$

If $RRE((\hat{\boldsymbol{\alpha\beta}}), (\tilde{\boldsymbol{\alpha\beta}})) > 1$ $(<)$, $(\hat{\boldsymbol{\alpha\beta}})$ is better (worse) than $(\tilde{\boldsymbol{\alpha\beta}})$. We compare the risks, based on quadratic loss, of the robust estimators and LSE's due to the Monte Carlo simulation. We limited attention to $n = 5, 10$ and $I = J = 3$. The underlying distribution chosen here were contaminated normal; $0.9\Phi(x) + 0.1\Phi(x/\sqrt{10})$, the mixture of the normal and outlier; $0.95\Phi(x) + 0.05 I_{[10,\infty)}(x)$, standard normal, logistic with variance 1, double exponential with variance 1 and lognormal. For each setting, $2,000$ replications were used.

From Table 1, we can see that the proposed estimators are more efficient than the LSE's except the case that an undelying distribution is normal. Especially the proposed estimators are fairly efficient for the asymmetric distributions.

Table 1: The values for the relative risk efficiency of proposed estimators with respect to LSE's

$F(x)$	$N = 5$			$N = 10$		
	$\boldsymbol{\beta}$	$(\boldsymbol{\alpha\beta})$	$\boldsymbol{\nu}$	$\boldsymbol{\beta}$	$(\boldsymbol{\alpha\beta})$	$\boldsymbol{\nu}$
$\Phi(x)$	0.94	0.93	0.95	0.93	0.92	0.94
$0.9\Phi(x) + 0.1\Phi(x/3)$	1.29	1.27	1.25	1.34	1.34	1.34
$0.95\Phi(x) + 0.05I_{[5,\infty)}(x)$	1.33	1.33	1.92	1.49	1.50	2.94
logistic	1.04	1.05	1.03	1.07	1.07	1.09
double exponential	1.21	1.23	1.20	1.30	1.30	1.32
lognormal	1.84	1.82	0.94	2.38	2.40	0.46

Acknowledgement: The author is grateful to the referee for valuable comments.

REFERENCES

[1] Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.,* **35**, 73-101.

[2] Huber, P. J. (1981). *Robust Statistics*, Wiley-Intersciences.

[3] Jurečková, J. (1971). Nonparametric estimate of regression coefficients. *Ann. Math. Statist.,* **35**, 73-101.

[4] Jurečková, J. and Sen, P. K. (1996). *Robust Statistical Procedures: Asymptotics and Interrelations,* New York: John Wiley.

[5] Koenker, R. and Portnoy, S. (1990). M-estimation of multivariate regressions. *J. Amer. Statist. Assoc.,* **85**, 1060-1068.

[6] Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics,* New York: John Wiley.

[7] Shiraishi, T. (1989). R-estimators and confidence regions for treatment effects in multi-response experiments. *J. Multivariate Anal.,* **31**, 30-39.

[8] Shiraishi, T. (1991). Statistical inference based on aligned ranks for two-way manova with interaction. *Ann. Inst. Statist. Math.,* **43**, 715-734.

[9] Shiraishi, T. (1996). On scale-invariant M-statistics in multivariate k samples. *J. Japan Statist. Soc.,* **26**, 241-253.

[10] Singer, J. M. and Sen, P. K. (1985). M-methods in multivariate linear models. *J. Multivariate Anal.,* **17**, 168-184.

[11] Wiens, D. (1986). Minimax variance M-estimators of location in Kolmogorov neibourhoods. *Ann. Statist.,* **14**, 724-732.