On Measuring Dynamic Financial Risk and
SSAR Risk Management [*]

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Abstract

We discuss the problem of measuring financial risk and propose to use some dynamic risk measures when the underlying assets prices follow the continuous diffusion processes and the non-linear discrete time series. We shall develop a general procedure to estimate the dynamic conditional tail expectation approximately by using the asymptotic expansion approach via the Malliavin-Watanabe Calculus. Then we discuss some statistical problems of measuring dynamic risk from a set of discrete time series data.

Key Words

Financial Risk Measures, Dynamic VaR, Dynamic Tail Conditional Expectation, Approximate DTCE, Asymptotic Expansion Approach, SSAR Risk Management

1 Introduction

In the past decade the statistical method of measuring financial risk has been developed and extensively used in finance. It has been not only important in financial industries and their regulations, but also interesting in academic point of view on the statistical analysis of financial risk. Among several methods proposed including credit risk and others, the method of VaR (Value at Risk) has been the most important one partly because the BIS and central banks in many advanced countries have adopted the guidlines based on the VaR method for regulating the banking sector. See J.P. Morgan (1996) and Jorion (2000) on the details of the standard statistical as well as practical aspects of the VaR method in financial industries, for instance.

The main purpose of this study is to reconsider one important issue of measuring financial risk in the dynamical portfolio managemants and the regulations on financial risk control. The standard practice of measuing financial risk has been often static in the sense that the risk measures proposed are those at a paticular given period. In practice, however, the private financial corporations are conducting business in certain horizon of investments including asset allocations, derivative tradings and portfolio managements. Hence apparently there are some basic questions whether the well-known practical methods in financial industries and regulatory authorities are valid in the actual dynamic financial markets. We shall discuss some aspects of dynamic risk measures which could be different from the static ones and propose a new procedure of measuring dynamical financial risks in this respect. We are trying to propose some financial risk measures and risk control method which can be consistent in the dynamical point of view.

In the general case when the underlying asset prices follow a set of multidimensional diffusion processes, however, it is quite difficult to estimate the dynamic financial risk measures in the exact sense. It is partly because the distributions of the underlying assets of portfolios and their sample paths are quite complicated and the volatility of asset prices should not be treated as constants over time for practical point of view. In addition to these aspects, the discount rates such as the spot interest rates for evaluating the asset prices in the future dates should not be constant and their movements over time should be treated at the same time. In order to handle this general situation, we shall develop a procedure to estimate the dynamic risk measures approximately by using the asymptotic expansion approach, which has been developed by Kunitomo and Takahashi (1998, 2001), and Kunitomo and Kim (2001). We shall give some explicit formulae which are useful for calculating dynamical financial risk. Our method is based on the Malliavin-Watanabe Calculus, which is a powerful theory as the infinite dimensional stochastic analysis, and Yoshida (1992) was a pioneering work in this field

of applications.

The second purpose of this study is to develop the statsitical risk management method, which is called the SSAR (simultaneous swiching autoregressive) method. In the statistical analysis of financial data, there have been some evidences that the non-stationarity, time dependence, non-Gaussianity, and asymmetry of asset distributions should not be ignored. We shall propose to handle these aspects at the samte time by using a relatively simple non-linear time series model called the class of SSAR models. The SSAR modeling has been developed by Kuniomo and Sato (2000, 2001) in the statistical non-linear time series analysis. In addition to the theoretical developments, we shall examine the conventional VaR methodology commonly used by conducting a set of similations and compare it with the SSAR modeling by investigating real examples.

2   Dynamic Risk Measures

We consider the general framework of measuring dynamic financial risks in financial maekets. Let

(2.1)
$$V(t) = \sum_{i=1}^{n} \pi^i(t) S_t^i$$

be the non-negative value of a portfolio at time t consisting of n assets with the prices S_t^i $(i = 1, \cdots, n)$ and $\pi^i(t)$ be the share of the i-th asset at t. We assume that the investment horizon of portfolio is finite $(0 \leq t \leq T)$ and the i-th asset price S_t^i follows the stochastic differential equation

(2.2)
$$dS_t^i = S_t^i[b_i(t, \omega)dt + \sum_{j=1}^{d} \sigma_j^i(t, \omega)dB_t^j] \quad (i = 1, \cdots, n),$$

where $b_i(t, \omega)$ are the drift terms, $\{\sigma_j^i(t, \omega)\}$ are the volatility functions, and B_t^j $(j = 1, \cdots, d)$ are the standard Brownian motions.

We assume that $n \geq 1$, $d \geq 1$ and the strategies $\pi^i(t)$ $(i = 1, \cdots, n)$ satisfy the self-financing condition

(2.3)
$$dV(t) = \sum_{i=1}^{n} \pi^i(t)dS_t^i .$$

Then by using (2.2) this equation can be re-written as

(2.4)
$$dV(t) = V(t)[b(t)dt + \sum_{j=1}^{d} \sigma_j(t)dB_t^j] ,$$

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where

(2.5)
$$b(t) = \sum_{i=1}^{n} b_i(t)[\pi^i(t)\frac{S_t^i}{V(t)}] ,$$

and

(2.6)
$$\sigma_j(t) = \sum_{i=1}^{n}[\frac{\pi^i(t)S_t^i}{V(t)}]\sigma_j^i(t) \ (j = 1, \cdots, d).$$

We give two examples of our formulation. The first one is the simplest case, but it highlights the problem of measuring dynamic financial risks. The second one is the familiar formulation of the financial derivative pricing in the Black-Scholes theory.

Example 1: We take $n = d = 1$ and $V(t) = S_t$. The price of the asset S_t follows the geometric Brownian motion defined by

(2.7)
$$dS_t = S_t[b(t)dt + \sigma(t)dB_t] ,$$

where $b(t)$ and $\sigma(t)$ are some deterministic functions of time with $t \in [0, T]$.

Example 2: We take $n = d + 1$ and the first asset is the safe asset whose price process follows

(2.8)
$$dS_t^1 = r(t)S_t^1 dt ,$$

where $r(t)$ is the instantaneous spot interest rate at time t. This case can be a special case when we take $b_1(t) = r(t)$, $\sigma_j^1(t) = 0 \ (j = 1, \cdots, d)$, and

$$\pi^1(t)S_t^1 = V(t) - \sum_{i=2}^{n} \pi^i(t)S_t^i .$$

This example is the extended Black-Scholes model which has been often used for pricing financial derivatives.

Now we are trying to measure the dynamic financial risks when the underlying price processes follow the continuous diffusions. We shall first extend the standard definition of the Value-at-Risk (VaR) concept into the one in the dynamic framework.

Definition 1 : The dynamic Value at Risk with $100\alpha\%$ $(DVaR_\alpha)$ is defined by $DVaR_\alpha = V(0) - A_\alpha$ such that

(2.9)
$$1 - \alpha = P\left(\omega| \min_{0 \le t \le T} \frac{V(t)}{A(t)} > 1\right) ,$$

where the floor function is given by $A(t) = A_\alpha e^{\int_0^t r(s)ds}$ and $r(s)$ is the discount rate at t.

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Here $V(0)$ is the initial value of portfolio and $V(0) - A_\alpha$ corresponds to the standard VaR value in the static setting if we have used $1 - \alpha = P(V(T) > A(T))$. We usually use the spot interest rate in the finanicial markets as the discount rate.

It is important to notice that contrary to the standard VaR formulation we have the situation such that there are some points of time $V(t) < A(t)$ and

$$(2.10) \qquad V(T) > A(T) = A_\alpha e^{\int_0^T r_s ds}$$

at the same time. Then we need to use some risk measures to control the financial risks in the dynamic setting. We define the dynamic TCE (tail conditional expectation) value or the expected shortfall as follows.

Definition 2 : The dynamic TCE_α with $100\%\alpha$ are defined by

$$(2.11) \qquad DTCE_\alpha = E[e^{-\int_0^T r(s)ds}[A(T) - V(T)]I(\{A(T) \geq V(T)\})|C_T]$$

and

$$(2.12) \qquad DTCE_\alpha^* = E[[V(0) - e^{-\int_0^T r(s)ds}V(T)]I(\{A(T) \geq V(T)\})|C_T] ,$$

where $I(\cdot)$ is the indicator function and the set C_T in the conditional expectation operator is defined by

$$(2.13) \qquad C_T = \{\omega| \min_{0 \leq t \leq T} \frac{V(t)}{A(t)} > 1\}^c .$$

The DTCE measures are natural generalizations of the standard TCE measures discussed by Artzner et. al. (1999) and Jaschke (2001) extensively. From our definition the $DTCE_\alpha$ can be rewritten as

$$DTCE_\alpha = A_\alpha - \frac{1}{\alpha}E[e^{-\int_0^T r(s)ds}V(T)I(\{A(T) \geq V(T)\})|C_T] ,$$

and then we have the relation between two dynamic risk measures as

$$(2.14) \qquad DTCE_\alpha^* = DTCE_\alpha + DVaR_\alpha .$$

We notice that if we take $V(0) > A_\alpha$ and the underlying market is complete, then $DTCE_\alpha = 0$ when we take the perfect hedging strategy in Example 2 as we have expected from the standard derivative pricing theory. However, if the underlying financial market is incomplete and we can not use the perfect hedging strategy, then $DTCE_\alpha$ should be positive for any $0 < \alpha < 1$.

Now we consider the simple situation of Example 1. We shall use the following Lemma, which is the first part of Corollary 2.1 of Kunitomo and Ikeda (1992). For the sake of completeness we shall state the modified version of Theorem 2.1 of Kunitomo and Ikeda (1992) in the Appendix.

Lemma 1 : *Let S_t be the continuous process satisfying (2.7) when $b(t) = b, \sigma(t) = \sigma$ and $r(s) = r$ (b, σ and r are real constants). Then*

$$(2.15) \qquad P\left(\min_{0 \le t \le T} \frac{V(t)}{A(t)} > 1 , S(T) \in I \subset [A(T), +\infty)\right)$$

$$= \int_I \{\phi(\ln y; \ln S_0 + (b - \frac{\sigma^2}{2})T, \sigma\sqrt{T})$$

$$- (\frac{A}{S_0})^{\frac{2(b-r-\sigma^2/2)}{\sigma^2}} \phi(\ln y; \ln \frac{A^2}{S_0} + (b - \frac{\sigma^2}{2})T, \sigma\sqrt{T})\} \frac{1}{y} dy ,$$

where $A(t) = Ae^{rt}$ and $\phi(z; c_1, c_2)$ is the density function of the normal distribution $N(c_1, c_2)$.

By using this Lemma, we can set A_α such that

$$(2.16) \qquad 1 - \alpha = P\left(\omega | \min_{0 \le t \le T} \frac{V(t)}{A(t)}\right)$$

$$= \int_{\ln A(T)}^{\infty} \{\phi(z; \ln S_0 + (b - \frac{\sigma^2}{2})T, \sigma\sqrt{T})$$

$$- (\frac{A}{S_0})^{\frac{2(b-r-\sigma^2/2)}{\sigma^2}} \phi(z; \ln \frac{A^2}{S_0} + (b - \frac{\sigma^2}{2})T, \sigma\sqrt{T})\} dz$$

$$= \Phi[\frac{\ln S_0 - \ln A_\alpha + (b - r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}]$$

$$- (\frac{A_\alpha}{S_0})^{\frac{2(b-r-\sigma^2/2)}{\sigma^2}} \Phi[\frac{-\ln S_0 + \ln A_\alpha + (b - r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}] ,$$

where $\Phi(\cdot)$ is the distribution function of the standard normal random variable. Then we have the following result.

Theorem 1 : *Let S_t be the continuous process satisfying (2.7) when $b(t) = b, \sigma(t) = \sigma$ and $r(s) = r$ (b, σ and r are real constants). Then the $DTCE_\alpha$ is given by*

$$(2.17) \qquad DTCE_\alpha = A_\alpha - \frac{1}{\alpha} S_0 e^{(b-r)T} \Phi[\frac{\ln(A_\alpha/S_0) - (b - r + \sigma^2/2)}{\sigma\sqrt{T}}] .$$

There has been an important concept of the coherent risk measures (CRM), which was introduced and discussed by Artzner et. al. (1999) and Jaschke (2001). (See Kusuoka (2000) for the related problem.)

Let $\rho(\cdot)$ be a CRM satisfying Axioms of Artzner et. al. (1999) except Axiom T. In the dynamic setting we are interested in we need to modify Axiom T of CRM as

$$(2.18) \qquad \rho(X(T) + c\, e^{\int_0^T r(s)ds}) = \rho(X(T)) - c$$

for any positive constant c, where $X(T)$ is a functional of the underlying Brownian motions in $[0, T]$ and $r(t)$ is the spot interest rate at t. When $r(s) = r$ for all s, it is obvious that $DTCE_\alpha$ we have introduced is one of CRM.

Except the special case of Example 1, it is difficult to obtain the explicit formulae of DTCE when the underlying asset prices follow the multi-dimensional diffusions in the general cases. Then in the next section we shall develop a method of estimating DTCE approximately by using the asymptotic expansion approach.

3   Asymptotic Expansion Method for Risk Measures

We shall develop the asymptotic expansion method called the small disturbance asymptotics for estimating the dynamic risk measures in the general case. For this purpose we introduce some notations in this section. Let $S_t^{(\delta)} = (S_t^{(\delta)i})$ $(i = 1, \cdots, n; 0 < t \leq T)$ be the prices of the underlying securities at t with a parameter $0 < \delta \leq 1$. We consider the situation that this security pays no dividends and the price process follows the stochastic differential equation :

$$(3.19) \qquad S_t^{(\delta)i} = S_0^i + \int_0^t S_s^{(\delta)} b_s^{(\delta)i} ds + \int_0^t S_s^{(\delta)i} \sum_{j=1}^d \sigma_s^{(\delta)ij} dB_{1s}^j \ ,$$

where $b_t^{(\delta)} = (b_t^{(\delta)i})$ are the drift coefficients, and $\sigma_t^{(\delta)ij}$ are the instantaneous volatility at t with the parameter $0 < \delta \leq 1$ and $B_{1t} = (B_{it}^j)$ are the set of d independent Brownian motions. We assume that the non-negative stochastic process $\sigma_s^{(\delta)ij}$ follows the stochastic differential equation :

$$(3.20) \qquad \sigma_s^{(\delta)ij} = \sigma_0^{ij} + \int_0^s \mu_\sigma^{ij}(\sigma_u^{(\delta)}, u, \delta) du + \delta \int_0^s w_\sigma^{ij}(\sigma_u^{(\delta)}, u) dB_{2u}^{ij} \ ,$$

where $B_{2t} = (B_{2t}^{ij})$ are the set of $n \times d$ Brownian motions. For the interest rate processes, we assume that there exists a locally riskless money market [1] and the money market account (accumulation factor) is given by $M^{(\epsilon)}(t) = \exp(\int_0^t r_s^{(\epsilon)} ds)$, where ϵ is a parameter with $0 < \epsilon \leq 1$. We assume that the non-negative (instantaneous) spot interest rate process $r_t^{(\epsilon)}$, which is consistent with the money market and the discount bond markets, follows the stochastic differential equation :

$$(3.21) \qquad r_s^{(\epsilon)} = r_0 + \int_0^s \mu_r(r_u^{(\epsilon)}, u, \epsilon) du + \epsilon \int_0^s w_r(r_u^{(\epsilon)}, u) dB_{3u} \ ,$$

where B_{3t} is the standard Brownian motion. In (3.19)-(3.21) we consider the general situation when three sets of Brownian motions are correlated and their instantaneous

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[1] Implicitly we are considering the situation when there also exist bond markets in the economy and let $P^{(\epsilon)}(s, t)$ $(0 \leq s \leq t \leq T)$ be the discount bond price at s with the maturity date t . As the simplest case we have the situation when all discount bond prices $P^{(\epsilon)}(s, t)$ $(0 \leq s \leq t \leq T)$ are solely determined by the single factor $\{r_t^{(\epsilon)}\}$. However, we can also formulate HJM term structure of interest rates model in which the spot interest rate is not necessarily a Markovian process. See Kunitomo and Takahashi (2001), and Section 5.2 of Kunitomo and Kim (2001) for the details.

correlations are given by

$$(3.22) \qquad E\left[\ d\mathbf{B}_t^{i,jk} \quad d\mathbf{B}_t^{i,jk\prime}\ \right] = \begin{pmatrix} 1 & \rho_\sigma^{i,jk} & \rho_r^i \\ \rho_\sigma^{i,jk} & 1 & \rho_{\sigma r}^{jk} \\ \rho_r^i & \rho_{r\sigma}^{jk} & 1 \end{pmatrix} dt\ ,$$

where we denote $d\mathbf{B}_t^{i,jk} = (dB_{1t}^i, dB_{2t}^{jk}, dB_{3t})'$.

By using the above notations we write the portfolio value process $V^{(\delta)}(t)$ as

$$(3.23) \qquad V^{(\delta)}(t) = \sum_{i=1}^n \pi_s^{(\epsilon,\delta)i} S_t^{(\delta)i}$$

and

$$(3.24) \qquad V^{(\delta)}(t) = V(0) + \int_0^t \sum_{i=1}^n \pi_s^{(\epsilon,\delta)i} dS_t^{(\delta)i} ds$$

where $\pi_t^{(\epsilon,\delta)i}$ is the quantity of asset $S_t^{(\delta)i}$ in the portfolio at t. We make the following assumption for the resulting simplicity of our analysis.

Assumption I : The drift functions $b_t^{(\delta)i}$ $(i = 1, \cdots, n)$ and relative share functions for the assets $i\ (= 1, \cdots, n)$

$$\theta_t^{(\epsilon,\delta)i} = \frac{\pi_t^{(\epsilon,\delta)i} S_t^{(\delta)i}}{V^{(\delta)}(t)}$$

are given by

$$(3.25) \qquad b_t^{(\delta)i} = b_t^i + o_p(\delta)\ ,\ \ \theta_t^{(\epsilon,\delta)i} = \theta_s^i + o_p(\epsilon,\delta)\ ,$$

where b_t^i and θ_s^i $(i = 1, \cdots, n)$ are the deterministic functions of time t.

We shall analyze the effects of the stochastic volatility and the stochastic interest rates on the financial risk measures when both ϵ and δ are small. In order to develop the asymptotic expansion approach when both δ and ϵ are small, we need to have some regularity conditions for that the solutions of (3.19)-(3.21) are well-behaved and the stochastic expansions of the stochastic processes $\{r_t^{(\epsilon)}\}$ and $\{\sigma_t^{(\delta)}\}$ can be allowed.

Assumption II : Let $S_t^{(\delta)}, \sigma_t^{(\delta)}$ and $r_s^{(\epsilon)}$ be the set of diffusion processes with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ which satisfy (3.19)-(3.21). (i) The drift functions $\mu_r(r_t^{(\epsilon)}, t, \epsilon)$, $\mu_\sigma^{ij}(\sigma_t^{(\delta)}, t, \delta)$ and the diffusion functions $w_r(r_t^{(\epsilon)}, t), w_\sigma^{ij}(\sigma_t^{(\delta)}, t)$ are $\mathcal{F}_t$-measurable, bounded, and Lipschitz continuous with respect to their first arguments. Also there exist a finite K_1 such that

$$(3.26) \qquad \int_0^T [r_s^{(\epsilon)} + \frac{1}{2}\|\sigma_s^{(\delta)}\|^2] ds\ \leq K_1\ .$$

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(ii) The drift functions are continuously twice differentiable and their first and second derivatives are bounded uniformly in ϵ and δ . The volatility functions are continuously differentiable and their first derivatives are bounded uniformly in ϵ and δ.

(iii) For any $0 < t \le T$ and any i $(= 1, \cdots, d)$ we have

$$(3.27) \qquad \int_0^t \sum_{i,i'=1}^n \sum_{j=1}^d \theta_s^i \theta_s^{i'} \; \sigma_s^{ij} \sigma_s^{i'j} ds > 0 \; ,$$

where σ_s^{ij} are the solution of the ordinary differential equation

$$(3.28) \qquad \sigma_t^{ij} = \sigma_0^{ij} + \int_0^t \mu_\sigma^{ij}(\sigma_s, s, 0) ds \; .$$

Assumption III : There exists a positive c $(0 < c < \infty)$ such that

$$(3.29) \qquad \lim_{\epsilon, \delta \to 0} \frac{\delta}{\epsilon} = c \; .$$

Under Assumption II we have the existence of the unique strong solution for $\{S_t^{(\epsilon,\delta)}\}$, $\{\sigma_t^{(\delta)}\}$ and $\{r_t^{(\epsilon)}\}$ satisfying the SDEs in (3.19)-(3.21) by using the standard results in Section IV of Ikeda and Watanabe (1989). The conditions in Assumptions I-III are quite strong and could be relaxed considerably. In any case, however, we need some conditions to assure the boundedness of risk measures. For the purpose of practical applications we need some approximation arguments to deal with the stochastic processes including the well-known non-negative interest rates and volatility processes.

In the rest of this section, we shall investigate the asymptotic behavior of the fortfolio value process in the situation when $\epsilon \downarrow 0$ and $\delta \downarrow 0$. We shall drive the explicit form of $V^{(\delta)}(t)$ and

$$(3.30) \qquad A^{(\epsilon)}(t) = A_\alpha e^{\int_0^t r_s^{(\epsilon)} ds}$$

for any $0 \le t \le T$ in the small disturbance asymptotic approach. Let

$$(3.31) \qquad D_r^{(\epsilon)}(t) = \frac{1}{\epsilon} [r_t^{(\epsilon)} - r_t] \; ,$$

where $r_t = r_t^{(0)}$ is the solution satisfying the ordinary differential equation

$$(3.32) \qquad r_t = r_0 + \int_0^t \mu_r(r_s, s, 0) ds \; .$$

By substituting $r_t^{(\epsilon)} = r_t + \epsilon D_r^{(\epsilon)}(t)$ into (3.32), we have

$$\epsilon D_r^{(\epsilon)}(t) \;\; = \;\; \int_0^t \left\{ [\mu_r(r_s + \epsilon D_r^{(\epsilon)}(s), s, \epsilon) - \mu_r(r_s, s, \epsilon)] + [\mu_r(r_s, s, \epsilon) - \mu_r(r_s, s, 0)] \right\} ds$$

$$(3.33) \qquad + \epsilon \int_0^t w_r(r_s + \epsilon D_r^{(\epsilon)}(s), s) dB_{3s} \; .$$

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Then by using Assumption II, we can find positive constants c_1 and c_2 such that for any t

$$(3.34) \qquad |D_r^{(\epsilon)}(t)| \leq \int_0^t [c_1|D_r^{(\epsilon)}(s)| + c_2]ds + |\int_0^t w_r(r_s + \epsilon D_r^{(\epsilon)}(s), s)dB_{3s}| .$$

Also by using the standard arguments in stochastic analysis, the martingale inequality, and the Grownwall inequality, we can find positive constants c_3 and c_4 such that

$$(3.35) \qquad E[|D_r^{(\epsilon)}(t)|^2] \leq c_3 e^{c_4 t} , E[\sup_{0 \leq t \leq T} |D_r^{(\epsilon)}(t)|^2] < +\infty$$

uniformly with respect to ϵ . Hence we confirm the convergence in probability that $r_t^{(\epsilon)} \to r_t$ uniformly with respect to t as $\epsilon \downarrow 0$.

Let

$$(3.36) \qquad E_r^{(\epsilon)}(t) = \frac{1}{\epsilon^2}[r_t^{(\epsilon)} - r_t - \epsilon D_r(t)] ,$$

where $D_r(t) = p\lim_{\epsilon \downarrow 0} D_r^{(\epsilon)}(t)$. Then by substituting $r_t^{(\epsilon)} = r_t + \epsilon D_r(t) + \epsilon^2 E_r^{(\epsilon)}(t)$ into (3.21), we can use a similar argument recursively to lead that $E[|E_r^{(\epsilon)}(t)|^2]$ is bounded uniformly with respect to t and ϵ and we have the uniform convergence of $D_r^{(\epsilon)}(t)$ to $D_r(t)$ with respect to t as $\epsilon \downarrow 0$ in probability. We need similar arguments on the existence and convergence of random variables $D_r(t)$, which we have omitted. By using the above arguments under Assumption I, the stochastic expansion of the instantaneous interest rate $r_t^{(\epsilon)}$ can be expressed by

$$(3.37) \qquad r_t^{(\epsilon)} = r_t + \epsilon\, D_r(t) + R_1$$

as $\epsilon \downarrow 0$, where the remainder term R_1 is in the order $o_p(\epsilon)$. Then by using (3.33) and convergence arguments of its each terms, $D_r(t)$ can be regarded as the solution of the stochastic differential equation :

$$(3.38) \qquad D_r(t) = \int_0^t [\partial\mu_r(r_s, s, 0)D_r(s) + \partial^\epsilon\mu_r(r_s, s, 0)]ds + \int_0^t w_r(r_s, s)dB_{3s} ,$$

where we denote

$$(3.39) \qquad \partial\mu_r(r_s, s, 0) = \frac{\partial\mu_r(r_s^{(\epsilon)}, s, \epsilon)}{\partial r_s^{(\epsilon)}}\bigg|_{r_s^{(\epsilon)}=r_s,\epsilon=0} ,$$

and

$$(3.40) \qquad \partial^\epsilon\mu_r(r_s, s, 0) = \frac{\partial\mu_r(r_s^{(\epsilon)}, s, \epsilon)}{\partial\epsilon}\bigg|_{\epsilon=0} .$$

In order to have a concise representation for $D_r(t)$, let Y_t^r be the solution of $dY_t^r = \partial\mu_r(r_t, t, 0)Y_t^r dt$ with the initial condition $Y_0^r = 1$. Then (3.38) can be solved as

$$(3.41) \qquad D_r(t) = \int_0^t Y_t^r (Y_s^r)^{-1}[w_r(r_s, s)dB_{3s} + \partial^\epsilon\mu_r(r_s, s, 0)ds] .$$

Similarly, under Assumptions II and III we can expand the integral equation (3.20) with respect to δ. By using the same argument as $r_t^{(\epsilon)}$, the stochastic expansion of the stochastic volatility $\sigma_t^{(\delta)ij}$ can be also expressed by

$$(3.42) \qquad \sigma_t^{(\delta)ij} = \sigma_t^{ij} + \delta\ D_\sigma^{ij}(t) + R_2$$

as $\delta \downarrow 0$, where the leading term σ_t^{ij} are the solution of the ordinary differential equation (3.23), the second term is given by $D_\sigma^{ij}(t) = p\lim_{\delta\downarrow 0} D_\sigma^{(\delta)ij}(t)$ with $D_\sigma^{(\delta)}(t)(= [\sigma_t^{(\delta)} - \sigma_t]/\delta)$, and the remainder term R_2 is of the order $o_p(\delta)$. Let $Y_t^{(\sigma)ij}$ be the solution of $dY_t^{(\sigma)ij} = \partial\mu_\sigma^{ij}(\sigma_t, t, 0)Y_t^{\sigma ij}dt$ with the initial condition $Y_0^{(\sigma)ij} = 1$. Then because $D_\sigma^{ij}(t)$ is the solution of the corresponding stochastic differential equation as (3.38) for $\{\sigma_t^{(\delta)}\}$, we can express $D_\sigma^{ij}(t)$ as

$$(3.43) \qquad D_\sigma^{ij}(t) = \int_0^t Y_t^{(\sigma)ij}(Y_s^{(\sigma)ij})^{-1}[w_\sigma^{ij}(\sigma_s, s)dB_{2s}^{ij} + \partial^\delta\mu_\sigma^{ij}(\sigma_s, s, 0)ds]\ ,$$

where $\partial w_\sigma^{ij}(\sigma_s, s, 0)$ and $\partial^\delta\mu_\sigma^{ij}(\sigma_s, s, 0)$ are defined in the same ways as (3.39) and (3.40). Now we summarize the asymptotic behavior of the discounted portfolio process $\frac{V^{(\delta)}(t)}{A^{(\epsilon)}(t)}$ as $\epsilon \downarrow 0$ and $\delta \downarrow 0$ in the next proposition. The proof is similar to Section 6 of Kunitomo and Kim (2001).

Lemma 2 : (i) Under Assumptions I, II and III,

$$(3.44) \qquad \sup_{0 \le s \le t \le T} |\frac{V^{(\delta)}(t)}{A^{(\epsilon)}(t)} - \frac{V(t)}{A(t)}| \to 0\ (a.s.),$$

and

$$(3.45) \qquad \frac{V(t)}{A(t)} = \frac{V(0)}{A_\alpha}\exp\{\int_0^t \sum_{i=1}^n \sum_{j=1}^d \theta_s^i \sigma_s^{ij}dB_{1s}^j + \int_0^t \sum_{i=1}^n \mu_s ds - \frac{1}{2}\sigma(t)^2\}$$

as $\epsilon \downarrow 0$ and $\delta \downarrow 0$, where $\mu_s = \sum_{i=1}^n \theta_s^i b_s^i - r_s$, $\sigma(t)^2 = \int_0^t \sum_{i,i'=1}^n \sum_{j=1}^d \theta_s^i \theta_s^{i'} \sigma_s^{ij}\sigma_s^{i'j}ds$, and $\{r_t\}$ and $\{\sigma_t\}$ are the solutions of the ordinary differential equations (3.32) and (3.28).

(ii) Let $\sigma_{MC}[V^{(\delta)}(t)]$ be the Malliavin covariance of the discounted portfolio value process and $V_t^{(\delta)} \in \mathbf{D}^\infty(\mathbf{R})$ [2]. Then we have

$$(3.46) \qquad \sup_{0 \le s \le t \le T} |\sigma_{MC}[V^{(\delta)}(s)] - V_s^2 \int_0^s \sigma_u^2 du| \to 0\ (a.s.)$$

as $\delta \downarrow 0$.

---

[2] Let the H-differentiation be defined by $DF_h(\mathbf{w}) = \lim_{\epsilon\to 0}(1/\epsilon)[F(\mathbf{w}+\epsilon\mathbf{h}) - F(\mathbf{w})]$ for a Wiener functional $F(\mathbf{w})$ and $\mathbf{h} \in \mathbf{M}$, where $\mathbf{M}$ is the Cameron-Martin subspace of the squared integrable functions in the Wiener space $\mathbf{W}$. Then the Malliavin covariance is given by $\sigma_{MC}(F) =< DF(\mathbf{w}), DF(\mathbf{w}) >_H$, where $< \cdot >_H$ is the inner product of $\mathbf{M}$ space. We need some stronger conditions than Assumption I for the class $\mathbf{D}^\infty(\mathbf{R})$. See *Theorem 3.1* of Kunitomo and Takahashi (1998) for the details.

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Next we shall derive the asymptotic expansion of the discounted portfolio value process $e^{-\int_0^t r_s^{(\epsilon)}ds}V^{(\delta)}(t)$ as $\epsilon \downarrow 0$ and $\delta \downarrow 0$. If we set the leading term as

$$(3.47) \qquad X_t = \int_0^t \sum_{i=1}^n \theta_s^i \sum_{j=1}^n \sigma_s^{ij} dB_{1s}^j \,,$$

we can write

$$(3.48) \quad e^{-\int_0^t r_s^{(\epsilon)}ds}V_t^{(\delta)} = V(0)\exp\bigg\{ \bigg[X_t + \int_0^t \mu_s ds - \frac{\sigma(t)^2}{2} \bigg]$$
$$-\epsilon \int_0^t D_r(s)ds + \delta \bigg[\int_0^t \sum_{i=1}^n \sum_{j=1}^d D_\sigma^{ij}(s)dB_{1s}^j - \int_0^t \sum_{i=1}^n \sum_{j=1}^d \sigma_s^{ij} D_\sigma^{ij}(s)ds \bigg] + R_3 \bigg\} \,,$$

and

$$(3.49) \qquad \exp\bigg[\int_0^T r_t^{(\epsilon)}dt \bigg] = \exp\bigg\{ \int_0^T r_s ds + \epsilon \int_0^T D_r(s)ds + R_4 \bigg\} \,,$$

where R_3 and R_4 are the remaining terms of higher orders. Then we can obtain a stochastic expansion of the value process of the discounted portfolio at time t with respect to ϵ and δ which can be summarized in the next lemma.

Lemma 3 : *Under Assumptions I,II and III, an asymptotic expansion of the price process of the security $S_t^{(\epsilon,\delta)}$ at any particular time point t as $\epsilon \to 0$ and $\delta \to 0$ is given by*

$$e^{-\int_0^t r_s^{(\epsilon)}ds}V^{(\delta)}(t) \;=\; V(0)\exp\bigg\{ \bigg[X_t + \int_0^t \mu_s ds - \frac{\sigma(s)^2}{2} \bigg]$$
$$+\;\; \epsilon\big[\frac{\Sigma_{12}^{(r)}(t)}{\sigma(t)^2}X_{1t} + \lambda_r(t)\big] + \delta\Sigma_{12}^{(\sigma)}(t)\bigg[\frac{X_{1t}^2}{\sigma(t)^4} - \frac{X_{1t}}{\sigma(t)^2} - \frac{1}{\sigma(t)^2} \bigg]$$
$$(3.50) \qquad +\;\; \delta\lambda_\sigma(t)[\frac{X_{1t}}{\sigma(t)^2} - 1] + R_5 \bigg\} \,,$$

where we use the notations

$$\Sigma_{12}^{(r)}(t) \;=\; (-1)\int_0^t \bigg(\int_u^t Y_s^r ds \bigg) (Y_u^r)^{-1} w_r(r_u, u, 0) \sum_{i=1}^n \sum_{j=1}^d \theta_u^i \sigma_u^{ij} \rho_r^j du \,,$$

$$\lambda_r(t) \;=\; (-1)\int_0^t \bigg(\int_u^t Y_s^r ds \bigg) (Y_u^r)^{-1} \partial^\epsilon \mu_r(r_u, u, 0) du \,,$$

$$\Sigma_{12}^{(\sigma)}(t) \;=\; \int_0^t \bigg(\int_u^t \sum_{i,i'=1}^n \theta_s^i \theta_s^{i'} \sum_{j=1}^d \sigma_s^{ij} Y_s^{(\sigma)ij} ds \bigg) (Y_u^{(\sigma)ij})^{-1} w_\sigma^{ij}(\sigma_u, u) \sigma_s^{i'j} \rho_\sigma^j du \,,$$

$$\lambda_\sigma(t) \;=\; \int_0^t \bigg(\int_u^t \sum_{i,i'=1}^n \sum_{j=1}^d \theta_s^i \theta_s^{i'} \sigma_s^{i'j} Y_s^{(\sigma)ij} ds \bigg) (Y_u^{(\sigma)ij})^{-1} \partial^\delta \mu_\sigma^{ij}(\sigma_u, u, 0) du$$

and R_5 is the remainder term of the order $o_p(\epsilon, \delta)$.

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Under Assumptions I and II we can find positive constant c_5 and c_6 such that

$$(3.51) \qquad E[V^{(\delta)}(t)] \leq c_5 e^{c_6 t} \;,\; E[\sup_{0 \leq t \leq T} |V^{(\delta)}(t)|^p] < +\infty$$

uniformly with respect to δ for any $p \geq 2$. Also it is straightforward to show that $E[\exp(c_7(\delta)\|X_t\|^2))]$ is bounded for sufficiently small $c_7(\delta)$ and $E[\exp\{\int_0^T \sigma_s^{(\delta)ij} dB_{1s}^j\}]$ are bounded for any δ and j $(j = 1, \cdots, d)$. (See Section IV of Ikeda and Watanabe (1989).)

We consider the asymptotic expansion of the theoretical value of the dynamic conditional tail expectation (DTCE). We have found that this problem is quite similar to the valuation problem of European put options developed by Section 3.2 of Kunitomo and Kim (2001). We write the second term of the value of DTCE as

$$(3.52) \qquad DTCE_\alpha = E\left[Z_T^{(\epsilon,\delta)} I(A(T)^{(\epsilon)} \geq V^{(\delta)}(T)) | C_T\right]$$

where

$$Z_T^{(\epsilon,\delta)} = \exp\left(-\int_0^T r_t^{(\epsilon)} ds\right) V^{(\delta)}(T) \;.$$

By substituting the relations of Lemma 3 into $Z_T^{(\epsilon,\delta)}$, we can obtain the expression for $Z_T^{(\epsilon,\delta)}$ as

$$(3.53) \qquad Z_T^{(\epsilon,\delta)} = Z_0 + \delta Z_1^\delta + \epsilon Z_1^\epsilon + R_6$$

where

$$Z_0 = V(0)\exp\left(X_T + \int_0^T \mu_s ds - \frac{1}{2}\sigma^2(T)\right) \;,$$

$$Z_1^\delta = Z_0 \times \left[\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^2}\left(\frac{X_{1T}^2}{\sigma(T)^2} - X_{1T} - 1\right) + \lambda_\sigma(T)\left(\frac{X_{1T}}{\sigma(T)^2} - 1\right)\right] \;,$$

$$Z_1^\epsilon = Z_0 \times \left\{\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2}X_{1T} + \lambda_r(T)\right\} \;,$$

and R_6 is the remainder term of the order $o_p(\epsilon,\delta)$. Hence we have reduced our problem into the evaluation of the expectations as

$$\begin{aligned} DTCE_\alpha &= E[Z_0 I(A^{(\epsilon)}(T) \geq V^{(\delta)}(T))] + E[Z_1^\delta I(A^{(\epsilon)}(T) \geq V^{(\delta)}(T))] \\ &+ E[Z_1^\epsilon I(A^{(\epsilon)}(T) \geq V^{(\delta)}(T))] + E[R_7 I(A^{(\epsilon)}(T) \geq V^{(\delta)}(T))] \;, \end{aligned}$$

where $I(\,\cdot\,)$ is the indicator function. By the result of lengthy derivations as Section 6.3 of Kunitomo and Kim (2001), we finally have obtained the theoretical value of DTCE as the next theorem.

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Theorem 2 : *Under Assumptions I,II and III, an asymptotic expansion of the theoretical value of the DTCE when the interest rate and volatility are stochastic, is given by*

$$
\begin{aligned}
DTCE_\alpha &= \left[A_\alpha - \frac{V(0)}{\alpha} e^{\int_0^T \mu_s ds} \Phi(-d_1) \right] \\
&+ \epsilon \frac{V(0)}{\alpha} e^{\int_0^T \mu_s ds} \{ [\frac{\Sigma_{12}^{(r)}(T)}{\sigma(T)^2}(-d_2 + \sigma(T)) + \frac{\lambda_r(T)}{\sigma(T)}]\phi(d_1) - [\Sigma_{12}^{(r)}(T) + \lambda_r(T)]\Phi(-d_1)\} \\
&+ \delta \frac{V(0)}{\alpha} e^{\int_0^T \mu_s ds} [\frac{\Sigma_{12}^{(\sigma)}(T)}{\sigma(T)^3}(d_2^2 - 1) - \frac{\lambda_\sigma(T)}{\sigma(T)^2}d_2]\phi(d_1) + o(\epsilon, \delta)
\end{aligned}
$$

as $\epsilon, \delta \downarrow 0$, where $\Phi(\cdot)$ is the distribution function of the standard normal variable and $\phi(\cdot)$ is its density function, $d_2 = d_1 - \sigma(T)$, and

$$
(3.54) \qquad d_1 = \frac{1}{\sigma(T)} \left[\log \frac{V(0)}{A_\alpha} + \int_0^T \mu_s ds + \frac{1}{2}\sigma(T)^2 \right],
$$

and $\Sigma_{12}^{(r)}(T), \Sigma_{12}^{(\sigma)}(T), \lambda_r(T)$ and $\lambda_\sigma(T)$ are defined in Lemma 3.

Let the first term be $DTCE_0$ and we set the coefficients of ϵ and δ to be $DTCE_r$ and $DTCE_\sigma$, respectively. Then the DTCE value can be decomposed into the first term and the adjustment terms as

$$
(3.55) \qquad DTCE_\alpha = DTCE_0 + \epsilon \, DTCE_r + \delta \, DTCE_\sigma + o(\epsilon, \delta)
$$

where $DTCE_0$ stands for the DTCE value under the assumptions of constant interest rate and volatility. The second term represents the adjustment value induced by the deterministic interest rate which in itself relies on the assumed interest rate and volatility model. The third term and the fourth term are the adjustment values induced by the stochastic interest rate and the stochastic volatility, respectively. Hence our results include the result reported in Section 2 as special cases in the sense of the small disturbance asymptotics.

There are some further considerations needed in implementing our procedure in this section. First, in the general case it is not an easy task to estimate the $DVaR_\alpha$ value. One suggestion might be to calculate it by setting $\delta = \epsilon = 0$ and call it as the approximate DVaR value and then we can calculate the approximate DTCE value. Second, if we relax Assumption I and allow more complicated strategies of $\pi^{(\delta)i}$ ($i = 1, \cdots, n$), then Theorem 2 should be modified considerably.

4 The method of SSAR Risk Management

In this section we shall consider the method of measuring financial risk by using a class of the simultaneous switching autoregressive (SSAR) models, which has been developed

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by Kunitomo and Sato (2000, 2001). Because we shall use the discrete time setting, we denote y_t^i be the price of the i-th asset at time t.

Let

(4.56)
$$V_t = \sum_{i=1}^{n} \pi_t^i y_t^i$$

be the non-negative value ot a portfolio at time t consisting of n assets with the price y_t^i $(i = 1, \cdots, n)$ and π_t^i be the share of the i-th asset at t. We assume that the investment horizon of portfolio is finite $(0 \le t \le T)$ and the i-th asset price y_t^i follows a class of non-linear time series model

(4.57)
$$\Delta y_t^i = G_{\sigma^i}(y_{t-1}^i, \cdots, y_{t-p}^i, v_t^i, \cdots, v_{t-r}^i) \ (i = 1, \cdots, n),$$

where $\Delta y_t = y_t - y_{t-1}$, p and r are non-negative integers, $G_{\sigma^i}(\cdot)$ are non-linear function and $\{v_t^i\}$ are i.i.d. sequence of random variables with $E[v_t^i] = 0$ and $E[(v_t^i)^2] = 1$.

We assume that

(i) $\{y_t^i, t = 0, \pm 1, \cdots\}$ satisfy the stochastic difference equation

(4.58)
$$\Delta y_t^i = G_{\sigma^i}(r_0^i + \sum_{j=1}^{p} r_j^i y_{t-j}^i + v_t^i \sqrt{h_t^i}) ,$$

where $G_{\sigma^i}(\cdot)$ is a continuous (but not necessarily differentiable) function, r_j^i $(j = 0, 1, \cdots, p)$ are unknown parameters, and

(ii) h_t^i (≥ 1) are the volatility functions which are the $\mathcal{F}_{t-1}-$measurable functions and $\mathcal{F}_{t-1}$ is the $\sigma-$field generated by the random variables $\{y_s^i, v_s^i; s \le t - 1, i = 1, \cdots, n\}$. In our applications we shall use the ARCH model for $\{h_t^i\}$ which satisfies the stochastic difference equation

(4.59)
$$h_t^i = 1 + \sum_{j=1}^{r} \alpha_j^i h_{t-j}^i v_{t-j}^{i\,2} ,$$

where α_j^i $(i = 1, \cdots, n; j = 1, \cdots, r)$ are unknown parameters with $\alpha_j^i \ge 0$ and $\sum_{j=1}^{r} \alpha_j^i < 1$.

In particular we further assume that

(iii) $G_{\sigma^i}(x)$ is a strictly increasing function satisfying

(4.60)
$$\lim_{x \to \infty} \frac{G_{\sigma^i}(x)}{x} = \sigma_1^i > 0 ,$$

and

(4.61)
$$\lim_{x \to -\infty} \frac{G_{\sigma^i}(x)}{x} = \sigma_2^i > 0 ,$$

where σ^i is the vector of unknown transformation parameters including σ_j^i $(j = 1, 2)$ appeared in (4.60) and (4.61).

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In the above formulation (4.58) is slightly different from many nonlinear time series models including the threshold autoregressive (TAR) models developed by Tong (1990). We shall be mainly interested in the time series movements which can be quite different in the upward phase ($\Delta y_t^i \geq 0$) and the downward phase ($\Delta y_t^i < 0$). Then (4.58) can give a simple but rich way to represent the time series modelling with these two phases. Because the transformation function $G_{\sigma^i}(\cdot)$ has some unknown parameters and the random noise v_t^i at t has not been realized at time $t-1$, the phase (the upward phase or downward phase, for instance) at time t is not determined in advance at time $t-1$. Also we shall be mainly interested in the case when the transformation function $G_{\sigma^i}(\cdot)$ in (4.58) is not differentiable. See Kunitomo and Sato (2000, 2001) for the details of the SSAR models.

It is possible to deal with the more general case on the volatility function such as the Generalized ARCH models and the stochastic volatility models. See Bollerslev (1986), Hamilton (1994), and Harvey and Shephard (1996), for instance. However, then the estimation procedures become more complicated than the methods in Section 3.2, and the estimated results sometimes become unstable in our limited experiences and we shall not pursue these possibilities further.

In the discrete time setting we also restrict the portfolio strategies which are self-financing in the financial literatures. Hence we impose the condition that

(4.62)
$$\sum_{i=1}^{n} [\pi_{t-1}^i - \pi_t^i] y_{t-1}^i = 0 \ \forall t \in [1, T].$$

Then we have the representation of the portfolio at time T as

(4.63)
$$
\begin{aligned}
V_T &= V_0 + \sum_{s=1}^{T} \Delta V_s \\
&= V_0 + \sum_{s=1}^{T} \sum_{i=1}^{n} \pi_{s-1}^i \Delta y_s^i \,,
\end{aligned}
$$

where Δy_t^i are determined by the stochastic difference equation (3.18).

Hence we can define the $DVaR_\alpha$ and $DTCE_\alpha$ in the discrete time setting as in Section 2. We define the dynamic VaR value as follows.

Definition 3 : The dynamic Value at Risk with $100\%\alpha$ ($DVaR_\alpha$) are defined by $DVaR_\alpha = V_0 - A_\alpha$ such that

(4.64)
$$1 - \alpha = P\left(\omega \big| \min_{0 \leq t \leq T} \frac{V_t}{A(t)} > 1 \right)$$

where $A(t) = A_\alpha e^{\sum_{j=0}^{t-1} r_s}$.

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Here V_0 is the initial value of portfolio and $V_0 - A_\alpha$ corresponds to the standard VaR value in the static setting. We also define the dynamic TCE (tail conditional expectation) value or the expected shortfall as follows.

Definition 4 : The dynamic TCE_α with $100\%\alpha$ are defined by

(4.65)
$$DTCE_\alpha = E[e^{-\sum_{s=0}^{t} r_s}[A(T) - V_T]I(A(T) \geq V(T))|C_T]$$

and

(4.66)
$$DTCE_\alpha^* = E[V_0 - e^{-\sum_{s=0}^{t} r_s} V_T I(A(T) \geq V(T))|C_T] \ ,$$

where the set C_T in the conditional expectation operator is defined by

$$C_T = \{\min_{0 \leq t \leq T} \frac{V_t}{A_t} > 1\}^c \ .$$

5 Appendix

Mathematically there was an incorrect statement in Theorem 2.1 of Kunitomo and Ikeda (1992), from which our Lemma 1 can be derived. Although it does lead to the main results unchanged as in Kunitomo and Ikeda (1992), we give its corrected version because it has been often referred in the derivative pricing literatures and the financial industries.

Theorem 3 : *Let S_t follow the geometric Brownian motion given by*

(5.67)
$$dS_t = \mu S_t dt + \sigma S_t dW_t \ ,$$

where W_t stands for the one-dimensional standard Brownian motion, μ is the drift parameter, σ is the volatility parameter, and the initial condition $S(0) = S_0$. (We assume that μ and σ are constant.) Set an interval $I \subset [A_2(t), A_1(t)]$ with $A_1(t) = Be^{\delta_1 t}$ and $A_2(t) = Ae^{\delta_2 t}$ and define the probability

(5.68)
$$P_I = \left(\min_{0 \leq t \leq T} \frac{S(t)}{A_2(t)} > 1 \ , \ \max_{0 \leq r \leq T} \frac{S(t)}{A_1(t)} < 1 \ and \ S(T) \in I\right) \ .$$

Then we have

(5.69)
$$P_I = \int_I (\sum_{k=-\infty}^{+\infty} k_n(y)) \frac{dy}{y} \ ,$$

where

(5.70)
$$\begin{aligned}
k_n(y) &= (\frac{B^n}{A^n})^{c_{1n}} (\frac{A}{S_0})^{c_{2n}} \phi[\ln(y); \ln(S_0 \frac{B^{2n}}{A^n}) + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}] \\
&\quad - (\frac{A^{n+1}}{S_0 B^n})^{c_{3n}} \phi[\ln(y); \ln(\frac{A^{2n+2}}{B^{2n} S_0}) + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}] \ ,
\end{aligned}$$

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where

$$c_{1n} = 2\frac{[\mu - \delta_2 - n(\delta_1 - \delta_2)]}{\sigma^2} - 1$$

$$c_{2n} = 2n\frac{[\delta_1 - \delta_2]}{\sigma^2}$$

$$c_{3n} = 2\frac{[\mu - \delta_2 + n(\delta_1 - \delta_2)]}{\sigma^2} - 1 \ ,$$

and $\phi(y; c_1, c_2)$ is the density function of the normal distribution $N(c_1, c_2)$.

We note that in the original proof of Theorem 2.1 of Kunitomo and Ikeda (1992) there were some ambiguous statements on the use of stopping times τ_1 and τ_2. For instance, the notation by $P_1(T, y) = P(\tau_1 < \tau_2 < T | Y(T) = y)$ should be read as $P_1(T, y) = P(\tau_1 < \tau_2, \tau_1 < T | Y(T) = y)$. Once these ambiguous statements are modified properly, it is straightforward to derive the main results, which remain the same as they were stated.

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