On asymptotic estimation theory of change-point problems for time series regression models

Takayuki Shiohama and Masanobu Taniguchi

Osaka University

Abstract. It is important to detect the structural change in the trend of time series model. This paper addresses the problem of estimating change point in the trend of time series regression models with circular ARMA residuals. First we show the asymptotics of the likelihood ratio between contiguous hypotheses. Next we construct the maximum likelihood estimator (MLE) and Bayes estimator (BE) for unknown parameters including change point. Then it is shown that the proposed BE is asymptotically efficient, and that MLE is not so generally.

Key words: Change point, time series regression, asymptotic efficiency, Bayes estimator, maximum likelihood estimator.

1. Introduction

The change point problem for serially correlated data has been extensively studied in the literature. References on various time series models with change-point can be found in the book of Csörgő and Horvath (1997) and the review paper of Kokoszka and Leipus (2000).

Focusing on a change point in the mean of linear process, Bai (1994) derived the limiting distribution of a consistent change-point estimator by least squares method. Later Kokoszka and Leipus (1998) studied the consistency of CUSUM type estimators of mean shift for dependent observations. Their results include long-memory processes. For a spectral parameter change in Gaussian stationary process, Picard (1985) developed the problem of testing and estimation. Giraitis and Leipus (1990,1992) generalized Picard's results to the case when the process concerned is possibly non-Gaussian.

For a structural change in regression model, a number of authors studied the testing and estimation of change point. It is important to detect the structural change in economic time series because parameter instability is common in this field. For testing structural changes in regression models with long-memory errors, Hidalgo and Robinson (1996) explored a testing procedure with nonstochastic and stochastic regressors. Asymptotic properties of change-point estimator in linear regression models were obtained by Bai(1998), where the error process may include dependent and heteroskedastic observations.

Despite the large body of literature on estimating unknown change-point in time series models, the asymptotic efficiency has been rarely discussed. For the case of independent and identically distributed observations, Ritov (1990) obtained an asymptotically efficient estimator of change point in distribution by a Bayesian approach. Also the asymptotic efficiency of Bayes estimator for change-point was studied by Kutoyants (1994) for diffusion-type process. Dabye and Kutoyants (2001) showed consistency for change-point in a Poisson process when the model was misspecified.

The present paper develops the asymptotic theory of estimating unknown parameters in time series regression models with circular ARMA residuals. The model and the assumptions imposed are explained in Section 2. Also Section 2 discusses the fundamental asymptotics for the likelihood ratio process between contiguous hypotheses. Section 3 provides the asymptotics of the maximum likelihood estimator (MLE) and Bayes estimator (BE) for unknown parameters including change-point. Then it is shown that the BE is asymptotically efficient, and that the MLE is not so generally. Some numerical examples are given in Section 4. All the proofs are collected in Section 5.

Throughout this paper we use the following notations. A' denotes the transpose of a vector or matrix A and $\chi(\cdot)$ is the indicator function.

2. Asymptotics of likelihood ratio and some lemmas

Consider the following linear regression model

$$\begin{aligned} y_t &= \{\boldsymbol{\alpha}'\chi(t/n \le \tau) + \boldsymbol{\beta}'\chi(t/n > \tau)\}\boldsymbol{z}_t + u_t, \\ &= r_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) + u_t, \qquad \text{(say)}, \quad t = 1, \dots, n \end{aligned} \tag{2.1}$$

where $\boldsymbol{z}_t = (z_{t1}, \dots, z_{tq})'$ are observable regressors, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_q)'$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_q)'$ are unknown parameter vectors, and $\{u_t\}$ is a Gaussian circular ARMA process with spectral density $f(\lambda)$ and $E(u_t) = 0$. Here τ is an unknown change-point satisfying $0 < \tau < 1$ and $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \tau) \in \boldsymbol{\Theta} \subset \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}$.

Letting

$$a^n_{jk}(h) = \begin{cases} \displaystyle\sum_{t=1}^{n-h} z_{t+h,j} z_{tk}, & h = 0, 1, \dots \\ \displaystyle\sum_{t=1-h}^{n} z_{t+h,j} z_{tk}, & h = 0, -1, \dots, \end{cases}$$

we will make the following assumptions on the regressors $\{\boldsymbol{z}_t\}$, which are a sort of Grenander's conditions.

Assumption 2.1.

(G.1) $a^n_{ii}(0) = O(n), \quad i = 1, \dots, q, \quad \text{and} \displaystyle\sum_{t=l}^{l+\rho} z^2_{ti} = O(\rho)$ for any $(1 \le l \le n)$.

(G.2) $\lim_{n\to\infty} z^2_{n+1,i}/a^n_{ii}(0) = 0, \quad i = 1, \dots, q.$

(G.3) The limit

$$\lim_{n\to\infty} \frac{a^n_{ij}(h)}{n} = \rho_{ij}(h)$$

exists for every $i, j = 1, \ldots, q$ and $h = 0, \pm 1, \ldots$.

Let $\boldsymbol{R}(h) = \{\rho_{ij}(h); i, j = 1, \ldots, q\}$.

(G.4) $\boldsymbol{R}(0)$ is nonsingular.

From (G.3) there exists a Hermitian matrix function $\boldsymbol{M}(\lambda) = \{M_{ij}(\lambda); i, j = 1, \ldots, q\}$ with positive semidefinite increments such that

$$\boldsymbol{R}(h) = \int_{-\pi}^{\pi} e^{ih\lambda} d\boldsymbol{M}(\lambda). \tag{2.2}$$

Suppose that the stretch of series from model (1) $\boldsymbol{y}_n = (y_1, \cdots, y_n)'$ is available. Denote the covariance matrix of $\boldsymbol{u}_n = (u_1, \cdots, u_n)'$ by Σ_n, and let $\boldsymbol{t}_n = (r_1, \cdots, r_n)'$ with $r_t = r_t(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)$. Then the likelihood function based on $\boldsymbol{y}_n$ is given by

$$L_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) = \frac{1}{(2\pi)^{n/2}|\Sigma_n|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{y}_n - \boldsymbol{t}_n)'\Sigma_n^{-1}(\boldsymbol{y}_n - \boldsymbol{t}_n)\right]. \tag{2.3}$$

Since we assume that $\{u_t\}$ is a circular ARMA process, it is seen that Σ_n has the following representation

$$\Sigma_n = \boldsymbol{U}_n^* \mathrm{diag}\{2\pi f(\lambda_1), \cdots, 2\pi f(\lambda_n)\}\boldsymbol{U}_n$$

where $\boldsymbol{U}_n = \{n^{-1/2}\exp(2\pi its/n); t, s = 1, \ldots, n\}$ and $\lambda_k = 2\pi k/n$ (see Anderson (1977)). Write

$$F_n(\lambda_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} (y_t - r_t) e^{-it\lambda_k}.$$

Then the likelihood function (2.3) is rewritten as

$$L_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) = \frac{1}{(2\pi)^n \{\prod_{k=1}^n f(\lambda_k)\}^{1/2}} \exp\left[-\frac{1}{2}\sum_{k=1}^{n} f(\lambda_k)^{-1}|F_n(\lambda_k)|^2\right]. \tag{2.4}$$

Define the local sequence for the parameters:

$$\boldsymbol{\alpha}_n = \boldsymbol{\alpha} + n^{-1/2}\boldsymbol{a}, \quad \boldsymbol{\beta}_n = \boldsymbol{\beta} + n^{-1/2}\boldsymbol{b}, \quad \tau_n = \tau + n^{-1}\rho \tag{2.5}$$

where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^q$ and $\rho \in \mathbb{R}$. Under the local sequence (2.5) the likelihood ratio process is represented as

$$\begin{aligned} Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho) &= \frac{L_n(\boldsymbol{\alpha}_n, \boldsymbol{\beta}_n, \tau_n)}{L_n(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau)} \\ &= \exp\left[-\frac{1}{2\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k) + \overline{d_n(\lambda_k)}\;\overline{A(\lambda_k)}\right\}\right. \\ &\qquad \left. -\frac{1}{2n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right] \end{aligned} \tag{2.6}$$

where $d_n(\lambda_k) = (2\pi n)^{-1/2}\sum_{t=1}^{n} u_t e^{it\lambda_k}$ and $A(\lambda_k) = A_1 + A_2 + A_3$ with

$$A_1 = (2\pi f(\lambda_k))^{-1/2} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s e^{-is\lambda_k},$$

$$A_2 = -(2\pi n f(\lambda_k))^{-1/2} \sum_{s=1}^{[\tau n+\rho]} \boldsymbol{a}' \boldsymbol{z}_s e^{-is\lambda_k}$$

and

$$A_3 = -(2\pi n f(\lambda_k))^{-1/2} \sum_{s=[\tau n+\rho]+1}^{n} \boldsymbol{b}' \boldsymbol{z}_s e^{-is\lambda_k}.$$

Here note that $d_n(\lambda_k), k = 1, 2, \ldots$ are i.i.d. complex normal random variables with mean 0 and variance $f(\lambda_k)$ (c.f. Anderson (1977)). Henceforth we write the spectral representation of u_t by

$$u_t = \int_{-\pi}^{\pi} e^{it\lambda} dZ_u(\lambda). \tag{2.7}$$

The asymptotic distribution of $Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho)$ is given as follows.

Theorem 1. *Suppose that Assumption 2.1 holds. Then for all $(\boldsymbol{\alpha}', \boldsymbol{\beta}', \tau) \in \boldsymbol{\Theta}$, the log-likelihood ratio has the asymptotic representation*

$$\begin{aligned}
& \log Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho) \\
= \; & (\boldsymbol{\beta} - \boldsymbol{\alpha})' W_1 + \sqrt{\tau} \boldsymbol{a}' W_2 + \sqrt{1-\tau} \boldsymbol{b}' W_3 \\
& - \frac{1}{8\pi^2} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_{s+j} \boldsymbol{z}_s' (\boldsymbol{\beta} - \boldsymbol{\alpha}) \\
& - \frac{1}{4\pi} (\sqrt{\tau} \boldsymbol{a} + \sqrt{(1-\tau)} \boldsymbol{b})' \int_{-\pi}^{\pi} f(\lambda)^{-1} d\boldsymbol{M}(\lambda) (\sqrt{\tau} \boldsymbol{a} + \sqrt{(1-\tau)} \boldsymbol{b}) + o_p(1) \\
= \; & \log Z(\boldsymbol{a}, \boldsymbol{b}, \rho) + o_p(1), \qquad (say),
\end{aligned}$$

where

$$W_1 = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda} f(\lambda)^{-1} dZ_u(\lambda),$$

$$W_2 = \frac{1}{2\pi\sqrt{n\tau}} \int_{-\pi}^{\pi} \sum_{s=1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda} (1 + e^{in\lambda}) f(\lambda)^{-1} dZ_u(\lambda)$$

and

$$W_3 = \frac{1}{2\pi\sqrt{n(1-\tau)}} \int_{-\pi}^{\pi} \sum_{s=[\tau n+\rho]+1}^{n} \boldsymbol{z}_s e^{is\lambda} (1 + e^{-in\lambda}) f(\lambda)^{-1} dZ_u(\lambda).$$

Here W_1, W_2 and W_3 are asymptotically normal with mean $\mathbf{0}$ and covariance matrix V_1, V_2 and V_3, respectively, where

$$\begin{aligned} V_1 &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left|\sum_{s=[\tau n]+1}^{[\tau n+\rho]} \boldsymbol{z}_s e^{is\lambda}\right|^2 f(\lambda)^{-1} d\lambda, \\ V_2 &= V_3 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2 f(\lambda)^{-1} d\boldsymbol{M}(\lambda). \end{aligned}$$

Next we present some fundamental lemmas which are useful in the estimation of change point.

Lemma 1. *Suppose that Assumption 2.1 holds. Then for any compact set $\mathscr{C} \subset \boldsymbol{\Theta}$, we have*

$$\sup_{\alpha,\beta,\tau\in\mathscr{C}} E_{\alpha,\beta,\tau} Z_n^{1/2}(\boldsymbol{a},\boldsymbol{b},\rho) \le \exp\{-g(\boldsymbol{a},\boldsymbol{b},\rho)\}$$

where

$$g(\boldsymbol{a},\boldsymbol{b},\rho) = (\boldsymbol{a}',\boldsymbol{b}')\boldsymbol{K}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} + c|\rho|$$

with some positive definite matrix $\boldsymbol{K}$ and $c > 0$.

Lemma 2. *Suppose that Assumption 2.1 holds. Then for any compact set $\mathscr{C} \subset \boldsymbol{\Theta}$, there exist $\kappa(\mathscr{C}) = \kappa, B(\mathscr{C}) = B$ such that*

$$\begin{aligned} &\sup_{(\alpha,\beta,\tau)\in\mathscr{C}|a_i|<H,|b_i|<H,\rho_j<H} [\|\boldsymbol{a}_1-\boldsymbol{a}_2\|^2 + \|\boldsymbol{b}_1-\boldsymbol{b}_2\|^2 + |\rho_1-\rho_2|^2]^{-1} \\ \times\quad & E_{\alpha,\beta,\tau}\left[Z_n^{1/4}(\boldsymbol{a}_2,\boldsymbol{b}_2,\rho_2) - Z_n^{1/4}(\boldsymbol{a}_1,\boldsymbol{b}_1,\rho_1)\right]^4 \le B(1+H^\kappa). \end{aligned}$$

3. Estimation theory

We are interested in the behavior of maximum likelihood estimator (MLE) and Bayes estimator (BE). To introduce there estimators, we need a loss function $w(y), y \in \mathbb{R}^d$ which is

1. nonnegative, continuous at point 0 and $w(0) = 0$, but is not identically 0;
2. symmetric: $w(y) = w(-y)$;
3. the sets $\{y : w(y) < c\}$ are convex for all $c > 0$.

We denote by $\boldsymbol{W}_p$ the class of loss functions satisfying 1-3 with polynomial majorants. The example of such function is $w(y) = |y|^p, p > 0$.

The MLE $\hat{\boldsymbol{\theta}}'_{ML} = (\hat{\boldsymbol{\alpha}}'_{ML}, \hat{\boldsymbol{\beta}}'_{ML}, \hat{\tau}_{ML})$ of $\boldsymbol{\theta}' = (\boldsymbol{\alpha}', \boldsymbol{\beta}', \tau)$ is defined by

$$L(\hat{\boldsymbol{\alpha}}_{ML}, \hat{\boldsymbol{\beta}}_{ML}, \hat{\tau}_{ML}) = \max_{(\alpha,\beta,\tau)\in\boldsymbol{\Theta}} L(\alpha, \beta, \tau) \tag{3.1}$$

The Bayes estimator $\tilde{\boldsymbol{\theta}}'_B = (\tilde{\boldsymbol{\alpha}}'_B, \tilde{\boldsymbol{\beta}}'_B, \tilde{\rho}_B)$ with respect to the quadratic loss function $l(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$ and a prior density $\pi(\cdot)$ is of the form

$$\tilde{\boldsymbol{\theta}}_B = \int_{\boldsymbol{\Theta}} \boldsymbol{\theta} p(\boldsymbol{\theta}|Y_n) d\boldsymbol{\theta} \tag{3.2}$$

where

$$p(\boldsymbol{\theta}|Y_n) = \frac{\pi(\boldsymbol{\theta}) L_n(\boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}} \pi(\boldsymbol{v}) L_n(\boldsymbol{v}) d\boldsymbol{v}}.$$

We suppose that the prior density is a bounded, positive and continuous function possessing a polynomial majorant on $\boldsymbol{\Theta}$. For $Z(\boldsymbol{u}), \boldsymbol{u} = (\boldsymbol{a}', \boldsymbol{b}', \rho)'$, in Theorem 1, define two random vectors $\hat{\boldsymbol{u}}' = (\hat{\boldsymbol{a}}', \hat{\boldsymbol{b}}', \hat{\rho})$ and $\tilde{\boldsymbol{u}}' = (\tilde{\boldsymbol{a}}', \tilde{\boldsymbol{b}}', \tilde{\rho})$ by relations

$$Z(\hat{\boldsymbol{u}}) = \sup_{\boldsymbol{u}\in\mathbb{R}^{2q+1}} Z(\boldsymbol{u}), \tag{3.3}$$

$$\tilde{\boldsymbol{u}} = \frac{\int_{\mathbb{R}^{2q+1}} \boldsymbol{u} Z(\boldsymbol{u}) d\boldsymbol{u}}{\int_{\mathbb{R}^{2q+1}} Z(\boldsymbol{v}) d\boldsymbol{v}}. \tag{3.4}$$

Theorem 2. *Let the parameter set* $\boldsymbol{\Theta}$ *be an open subset of* $\mathbb{R}^{2q+1}$. *Then the MLE is uniformly on* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \tau) \in \boldsymbol{\Theta}$, *consistent*

$$P - \lim_{n\to\infty} \hat{\boldsymbol{\theta}}_{ML} = \boldsymbol{\theta}$$

and converges in distribution

$$\mathscr{L}_\theta(\text{diag}\{\sqrt{n}, \cdots, \sqrt{n}, n\})(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta})\} \underset{d}{\longrightarrow} \mathscr{L}(\hat{\boldsymbol{u}}).$$

For any continuous loss function $w \in \boldsymbol{W}_p$, *we have*

$$\lim_{n\to\infty} E_{\boldsymbol{\theta}} w((\text{diag}\{\sqrt{n}, \cdots, \sqrt{n}, n\})(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta})) = Ew(\hat{\boldsymbol{u}}).$$

A similar theorem for Bayes estimators can be stated as follows.

Theorem 3. *The Bayes estimator* $\tilde{\boldsymbol{\theta}}_B$, *uniformly on* $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, *is consistent*

$$P_\theta - \lim_{n\to\infty} \tilde{\boldsymbol{\theta}}_B = \boldsymbol{\theta}$$

and converges in distribution

$$\mathscr{L}_\theta(\mathrm{diag}\{\sqrt{n},\cdots,\sqrt{n},n\})(\tilde{\boldsymbol{\theta}}_B-\boldsymbol{\theta})\}\underset{d}{\longrightarrow}\mathscr{L}(\tilde{\boldsymbol{u}}).$$

For any continuous loss function $w\in\boldsymbol{W}_p$*, we have*

$$\lim_{n\to\infty}E_{\boldsymbol{\theta}}w((\mathrm{diag}\{\sqrt{n},\cdots,\sqrt{n},n\})(\tilde{\boldsymbol{\theta}}_B-\boldsymbol{\theta}))=Ew(\tilde{\boldsymbol{u}}).$$

Remark. From Theorem 3 and Theorem 1.9.1 of Ibragimov and Has'minski(1981), we can see that the BE is asymptotically efficient such that

$$E\|\hat{\boldsymbol{u}}\|^2\geq E\|\tilde{\boldsymbol{u}}\|^2.$$

4. Numerical examples.

5. Real data analysis.

6. Proofs.

Proof of Theorem 1. From (2.6), we have

$$\begin{aligned}&\log Z_n(\boldsymbol{\alpha},\boldsymbol{\beta},\tau)\\=&-\frac{1}{2\sqrt{n}}\sum_{k=1}^n f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k)+\overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\}-\frac{1}{2n}\sum_{k=1}^n|A(\lambda_k)|^2\end{aligned}\quad(6.1)$$

First we evaluate the first term in (6.1). From (2.6) we have

$$\begin{aligned}&-\frac{1}{2\sqrt{n}}\sum_{k=1}^n f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k)+\overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\}\\=&-\frac{1}{2\sqrt{n}}\sum_{k=1}^n f(\lambda_k)^{-1/2}\\&\times\left\{d_n(\lambda_k)A_1+d_n(\lambda_k)A_2+d_n(\lambda_k)A_3+\overline{d_n(\lambda_k)A_1}+\overline{d_n(\lambda_k)A_2}+\overline{d_n(\lambda_k)A_3}\right\}\\=&\ E_1+E_2+E_3+E_4+E_5+E_6\qquad\text{(say)}.\end{aligned}$$

Write the spectral densiy $f(\lambda)$ in the form

$$f(\lambda)=\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}R_f(\lambda)e^{-ij\lambda}$$

where $R_f(j)$'s satisfy $\sum_{j=-\infty}^{\infty} |j|^m |R_f(j)| < \infty$ for any given $m \in \mathbb{N}$. Then, from Theorem 3.8.3 of Brillinger (1975) we may write

$$f(\lambda)^{-1} = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) e^{-ij\lambda}$$

where $\Gamma(j)$'s satisfy for any given $m \in \mathbb{N}$

$$\sum_{j=-\infty}^{\infty} |j|^m |\Gamma(j)| < \infty.$$

Then E_1 can be written as

$$\begin{aligned} E_1 &= -\frac{1}{2\sqrt{n}} \sum_{k=1}^{n} f(\lambda_k)^{-1/2} d_n(\lambda_k) A_1 \\ &= -\frac{1}{4n\pi} \sum_{k=1}^{n} f(\lambda_k)^{-1} \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s u_t e^{i(t-s)\lambda_k} \\ &= -\frac{1}{4n\pi} \sum_{k=1}^{n} \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) e^{-ij\lambda_k} \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s u_t e^{i(t-s)\lambda_k} \\ &= -\frac{1}{4n\pi} \frac{1}{2\pi} \sum_{k=1}^{n} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s u_t e^{i(t-s-j)\lambda_k} \end{aligned}$$

It is well known that

$$\sum_{k=1}^{n} e^{i(t-s-j)\lambda_k} = \begin{cases} n & \text{if } \; t-s-j = 0 \pmod{n} \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Since $-[\tau n + \rho] \le t - s \le [(1-\tau)n]$ and $\Gamma(j)$ satisfies $\sum_j |j|^k |\Gamma(j)| < \infty$ for any given m, we have

$$\sum_{|j| \ge n} |\Gamma(j)| \le \frac{1}{n^m} \sum_{|j| \ge n} (j)^m |\Gamma(j)| = o(n^{-m}).$$

Hence we have only to evaluate E_1 for $l = 0$ of $t - s - j = ln$. Thus E_1 is

$$\begin{aligned} E_1 &= -\frac{1}{4\pi} \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{t=1}^{n} \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s u_t \frac{1}{n} \sum_{k=1}^{n} e^{i(t-s-j)\lambda_k} \\ &\simeq -\frac{1}{8\pi^2} \sum_{j=-\infty}^{\infty} \Gamma(j) \sum_{s=[\tau n]+1}^{[\tau n+\rho]} (\boldsymbol{\beta} - \boldsymbol{\alpha})' \boldsymbol{z}_s \{u_{s+j}\} \equiv \tilde{E}_1 \quad \text{(say).} \end{aligned}$$

Then

$$
\begin{aligned}
\tilde{E}_1 &= -\frac{1}{8\pi^2}\sum_{j=-\infty}^{\infty}\Gamma(j)(\boldsymbol{\beta}-\boldsymbol{\alpha})'\sum_{s=[\tau n]+1}^{[\tau n+\rho]}\boldsymbol{z}_s\int_{-\pi}^{\pi}e^{ij\lambda}e^{is\lambda}dZ_u(\lambda) \\
&= -\frac{1}{4\pi}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\int_{-\pi}^{\pi}\sum_{s=[\tau n]+1}^{[\tau n+\rho]}\boldsymbol{z}_s e^{is\lambda}f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{\alpha})'W_1 \quad \text{(say)},
\end{aligned}
\tag{6.3}
$$

where $Z_u(\lambda)$ is the spectral measure of u_t defined by (2.7). Let $\sum_{s=[\tau n]+1}^{[\tau n+\rho]}\boldsymbol{z}_s e^{is\lambda} = \boldsymbol{A}(\lambda;h,\rho)$. we obeserve

$$
E(W_1W_1^*) \longrightarrow \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\boldsymbol{A}(\lambda;h,\rho)\boldsymbol{A}^*(\lambda;h,\rho)f(\lambda)^{-1}d\lambda \quad \text{as} \quad n\to\infty.
$$

Recalling that $\{u_t\}$ is Gaussian, we have

$$
W_1 \xrightarrow{D} N\left(\mathbf{0}, \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\boldsymbol{A}(\lambda;h,\rho)\boldsymbol{A}^*(\lambda;h,\rho)f(\lambda)^{-1}d\lambda\right) \tag{6.4}
$$

Similarly we obtain

$$
E_4 \sim \frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{\alpha})'W_1. \tag{6.5}
$$

Next we calculate the second term E_2 that is

$$
\begin{aligned}
E_2 &= -\frac{1}{2\sqrt{n}}\sum_{k=1}^{n}f(\lambda_k)^{-1/2}d_n(\lambda_k)A_2 \\
&= \frac{1}{4n\pi}\sum_{k=1}^{n}f(\lambda_k)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=1}^{[\tau n+\rho]}u_t\boldsymbol{a}'\boldsymbol{z}_s e^{i(t-s)\lambda_k} \\
&= \frac{1}{4n\pi}\frac{1}{2\pi}\sum_{k=1}^{n}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{a}'u_t\boldsymbol{z}_s e^{i(t-s)\lambda_k} \\
&= \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{a}'u_t\boldsymbol{z}_s\frac{1}{n}\sum_{k=1}^{n}e^{i\lambda_k(t-s-j)}.
\end{aligned}
$$

Here note that $n-1\geq t-s\geq -[\tau n]$. Because of (6.2) we have only to evaluate E_2 for $l=0,1$ of $t-s-j=ln$. Then

$$
E_2 \simeq \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{a}'}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}(u_{s+j}+u_{s+j+n})\boldsymbol{z}_s = \tilde{E}_2 \qquad \text{(say)}.
$$

Similarly as in $\tilde{E}_1$,

$$\begin{aligned}
\tilde{E}_2 &= \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{a}'}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\int_{-\pi}^{\pi}e^{is\lambda}e^{ij\lambda}(1+e^{in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \\
&= \frac{1}{4\pi}\frac{\boldsymbol{a}'}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\int_{-\pi}^{\pi}e^{is\lambda}\left(\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{is\lambda}\right)(1+e^{in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \\
&= \frac{\boldsymbol{a}'}{4\pi\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_s\int_{-\pi}^{\pi}e^{is\lambda}(1+e^{in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\boldsymbol{a}'}{4\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\sqrt{\tau}\boldsymbol{a}'}{2}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi\sqrt{n\tau}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\sqrt{\tau}\boldsymbol{a}'}{2}W_2 \qquad \text{(say)}, \qquad (6.6)
\end{aligned}$$

where

$$W_2 \xrightarrow{D} N\left(\boldsymbol{0}, \frac{1}{2\pi}\int_{-\pi}^{\pi}2f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\right), \qquad (6.7)$$

which follows from the Riemann-Lebesgue theorem and Grenander's conditions (G.1) - (G.4). Similarly we obtain.

$$E_5 \simeq \frac{\sqrt{\tau}\boldsymbol{a}'}{2}W_2. \qquad (6.8)$$

Next

$$\begin{aligned}
E_3 &= -\frac{1}{2\sqrt{n}}\sum_{k=1}^{n}f(\lambda_k)^{-1/2}d_n(\lambda_k)A_3 \\
&= \frac{1}{4n\pi}\sum_{k=1}^{n}f(\lambda_k)^{-1}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}u_t\boldsymbol{b}'\boldsymbol{z}_s e^{i(t-s)\lambda_k} \\
&= \frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'u_t\boldsymbol{z}_s e^{i(t-s)\lambda_k} \\
&= \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'u_t\boldsymbol{z}_s\frac{1}{n}\sum_{k=1}^{n}e^{i\lambda_k(t-s-j)}.
\end{aligned}$$

Since $[(1-\tau)n] \geq t-s \geq 1-n$, we have only to evaluate E_3 for $l = 0, -1$ of $t-s-j = ln$. Hence

$$E_3 \simeq \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{b}'}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}(u_{s+j}+u_{s+j-n})\boldsymbol{z}_s = \tilde{E}_3$$

Similarly as in $\tilde{E}_2$ we have

$$\begin{aligned}
\tilde{E}_3 &= \frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{\boldsymbol{b}'}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\int_{-\pi}^{\pi}e^{is\lambda}e^{ij\lambda}(1+e^{-in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \\
&= \frac{1}{4\pi}\frac{\boldsymbol{b}'}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\int_{-\pi}^{\pi}e^{is\lambda}\left(\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{is\lambda}\right)(1+e^{-in\lambda})dZ_u(\lambda)\boldsymbol{z}_s \\
&= \frac{\boldsymbol{b}'}{4\pi\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_s\int_{-\pi}^{\pi}e^{is\lambda}(1+e^{-in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\boldsymbol{b}'}{4\pi}\int_{-\pi}^{\pi}\left(\frac{1}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{-in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\sqrt{1-\tau}\boldsymbol{b}'}{2}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi\sqrt{n(1-\tau)}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_s e^{is\lambda}\right)(1+e^{-in\lambda})f(\lambda)^{-1}dZ_u(\lambda) \\
&= \frac{\sqrt{1-\tau}\boldsymbol{b}'}{2}W_3, \qquad (6.9)
\end{aligned}$$

where

$$W_3 \xrightarrow{D} N\left(\boldsymbol{0}, \frac{1}{2\pi}\int_{-\pi}^{\pi}2f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\right) \qquad (6.10)$$

Similarly we obtain.

$$E_6 = \frac{\sqrt{1-\tau}\boldsymbol{b}'}{2}W_3 \qquad (6.11)$$

Hence from (6.3), (6.5), (6.6), (6.8), (6.9) and (6.11), we have

$$\begin{aligned}
&-\frac{1}{2\sqrt{n}}\sum_{k=1}^{n}f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k)+\overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\} \\
\simeq\ &(\boldsymbol{\beta}-\boldsymbol{\alpha})'W_1+\sqrt{\tau}\boldsymbol{a}'W_2+\sqrt{1-\tau}\boldsymbol{b}'W_3. \qquad (6.12)
\end{aligned}$$

Next we evaluate the second term in (6.1), which is

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n}|A(\lambda_k)|^2 \\
= & -\frac{1}{2n}\sum_{k=1}^{n}(A_1+A_2+A_3)\overline{(A_1+A_2+A_3)} \\
= & -\frac{1}{2n}\sum_{k=1}^{n}(|A_1|^2+|A_2|^2+|A_3|^2+A_1\overline{A_2}+A_1\overline{A_3}+A_2\overline{A_3}++A_2\overline{A_1}+A_3\overline{A_1}+A_3\overline{A_2}).
\end{aligned}
$$

We have

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n}|A_1|^2 \\
= & -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{f(\lambda_k)}\left(\sum_{t=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t e^{it\lambda_k}\right)\left(\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
= & -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\sum_{t=[\tau n]+1}^{[\tau n+\rho]}\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_s'(\boldsymbol{\beta}-\boldsymbol{\alpha})e^{i(t-s)\lambda_k} \\
= & -\frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=[\tau n]+1}^{[\tau n+\rho]}\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_s'(\boldsymbol{\beta}-\boldsymbol{\alpha})\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k} \\
= & -\frac{1}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_{s+j}\boldsymbol{z}_s'(\boldsymbol{\beta}-\boldsymbol{\alpha}). \qquad (6.13)
\end{aligned}
$$

Next we have

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n}|A_2|^2 \\
= & -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda}\left(-\frac{1}{\sqrt{n}}\sum_{t=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_t e^{it\lambda_k}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
= & -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\sum_{t=1}^{[\tau n+\rho]}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{a}\left\{\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}\right\}.
\end{aligned}
$$

Note that $[\tau n] \geq t - s \geq -[\tau n]$. Similarly we have

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n}|A_2|^2 \\
\simeq\ & -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{a} = -\frac{\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\frac{1}{n\tau}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{a} \\
=\ & -\frac{\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\int_{-\pi}^{\pi}e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{a} = -\frac{\tau}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{a} \\
=\ & -\frac{\tau}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{a} \qquad (6.14)
\end{aligned}
$$

Also we obtain

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n}|A_3|^2 \\
=\ & -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{f(\lambda_k)}\left(-\frac{1}{\sqrt{n}}\sum_{t=[\tau n+\rho]+1}^{n}\boldsymbol{b}'\boldsymbol{z}_t e^{it\lambda_k}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
=\ & -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\sum_{t=[\tau n+\rho]+1}^{n}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{a}\left\{\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}\right\} \\
\simeq\ & -\frac{1}{4n\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{b}'\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{b} \\
=\ & -\frac{1-\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{b}'\frac{1}{n(1-\tau)}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{z}_{s+j}\boldsymbol{z}_s'\boldsymbol{b} \\
=\ & -\frac{1-\tau}{4\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{b}'\int_{-\pi}^{\pi}e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{b} \\
=\ & -\frac{1-\tau}{4\pi}\boldsymbol{b}'\int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{b} \\
=\ & -\frac{1-\tau}{4\pi}\boldsymbol{b}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{b}. \qquad (6.15)
\end{aligned}
$$

The fourth term becomes

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n} A_1\overline{A_2} \\
= & -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\left(\sum_{t=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t e^{it\lambda}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
= & \frac{1}{4\pi}\frac{1}{2\pi}\frac{1}{\sqrt{n}}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=h+1}^{[\tau n+\rho]}\sum_{s=1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{a}\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}
\end{aligned}
$$

From $1-\rho \le t-s \le [\tau n]+\rho-1$, $t-s-j=0$, it is seen that

$$
-\frac{1}{2n}\sum_{k=1}^{n} A_1\overline{A_2} \simeq \frac{1}{4\pi}\frac{1}{2\pi}\frac{1}{\sqrt{n}}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\boldsymbol{z}_{t-j}'\boldsymbol{a} = O\left(\frac{1}{\sqrt{n}}\right) \quad (6.16)
$$

Similarly we observe

$$
\frac{1}{2n}\sum_{k=1}^{n} A_1\overline{A_3} \simeq O(n^{-1/2}), \qquad \frac{1}{2n}\sum_{k=1}^{n} A_2\overline{A_1} \simeq O(n^{-1/2})
$$

$$
\text{and} \qquad \frac{1}{2n}\sum_{k=1}^{n} A_3\overline{A_1} \simeq O(n^{-1/2}). \quad (6.17)
$$

Now we evaluate

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n} A_2\overline{A_3} \\
= & -\frac{1}{4n\pi}\sum_{k=1}^{n}\frac{1}{2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)e^{-ij\lambda_k}\left(-\frac{1}{\sqrt{n}}\sum_{t=1}^{[\tau n+\rho]}\boldsymbol{a}'\boldsymbol{z}_t e^{it\lambda}\right)\left(-\frac{1}{\sqrt{n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{b}'\boldsymbol{z}_s e^{-is\lambda_k}\right) \\
= & -\frac{1}{4\pi}\frac{1}{2\pi}\frac{1}{n}\sum_{j=-\infty}^{\infty}\Gamma(j)\sum_{t=1}^{[\tau n+\rho]}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{a}'\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{b}\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k}.
\end{aligned}
$$

Since $-n+1 \le t-s \le -1$, we have only to evaluate for $t-s-j=0,-n$.

$$
\begin{aligned}
& -\frac{1}{2n}\sum_{k=1}^{n} A_2\overline{A_3} \\
\simeq & -\frac{1}{4\pi}\frac{1}{2\pi}\sqrt{\tau(1-\tau)}\sum_{j=-\infty}^{\infty}\Gamma(j)\frac{1}{\sqrt{\tau n}}\sum_{t=1}^{[\tau n+\rho]}\frac{1}{\sqrt{(1-\tau)n}}\sum_{s=[\tau n+\rho]+1}^{n}\boldsymbol{a}'\boldsymbol{z}_t\boldsymbol{z}_s'\boldsymbol{b}\frac{1}{n}\sum_{k=1}^{n}e^{i(t-s-j)\lambda_k} \\
\simeq & -\frac{\sqrt{\tau(1-\tau)}}{4\pi\cdot 2\pi}\sum_{j=-\infty}^{\infty}\Gamma(j)\boldsymbol{a}'\int_{-\pi}^{\pi}e^{ij\lambda}d\boldsymbol{M}(\lambda)\boldsymbol{b} \\
= & -\frac{\sqrt{\tau(1-\tau)}}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi}f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{b}. \quad (6.18)
\end{aligned}
$$

Similarly we have

$$-\frac{1}{2n}\sum_{k=1}^{n} A_3\overline{A_2} \ \simeq \ -\frac{\sqrt{\tau(1-\tau)}}{4\pi}\boldsymbol{a}'\int_{-\pi}^{\pi} f(\lambda)^{-1}d\boldsymbol{M}(\lambda)\boldsymbol{b}. \tag{6.19}$$

From the equations from (6.13) to (6.19) together with (6.4), (6.7), (6.10) and (6.12) complete the proof of theorem 1.

Proof of Lemma 1. From Hannan (1970) and Anderson (1977) the joint density of $d_n(\lambda_1),\cdots,d_n(\lambda_n)$ is given by

$$p(d_n(\lambda_1),\cdots,d_n(\lambda_n)) = C_n\prod_{k=1}^{k}\exp(-\overline{d_n(\lambda_k)}f(\lambda_k)^{-1}d_n(\lambda_k)) \tag{6.20}$$

where $C_n = \pi^{-n}\prod_{k=1}^{n} f(\lambda_k)^{-1}$. Using this,

$$\begin{aligned}
& EZ_n^{1/2}(\boldsymbol{a},\boldsymbol{b},\rho) \\
= \ & E\exp\left[-\frac{1}{4\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k)+\overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\}\right]\exp\left[-\frac{1}{4n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right] \\
= \ & \int\cdots\int C_n\exp\left(-\sum_{k=1}^{n}\overline{d_n(\lambda_k)}f(\lambda_k)^{-1}d_n(\lambda_k)\right) \\
& \times\exp\left(-\frac{1}{4\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\left\{d_n(\lambda_k)A(\lambda_k)+\overline{d_n(\lambda_k)}\ \overline{A(\lambda_k)}\right\}\right) \\
& \times\exp\left(-\frac{1}{4n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right)\boldsymbol{d}(d_n(\lambda_1)\cdots d_n(\lambda_n)) \\
= \ & \int\cdots\int C_n\exp\left[-\sum_{k=1}^{n}\left(f(\lambda_k)^{-1/2}d_n(\lambda_k)+\frac{\overline{A(\lambda_k)}}{4\sqrt{n}}\right)\left(\overline{f(\lambda_k)^{-1/2}d_n(\lambda_k)}+\frac{A(\lambda_k)}{4\sqrt{n}}\right)\right] \\
& \times\exp\left[\frac{1}{16n}\sum_{k=1}^{n}|A(\lambda_k)|^2-\frac{1}{4n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right]\boldsymbol{d}(d_n(\lambda_1)\cdots d_n(\lambda_n) \\
= \ & \exp\left(-\frac{3}{16n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right).
\end{aligned}$$

Recall that the definition of likelihood process in (2.6), we have

$$\exp\left(-\frac{3}{16n}\sum_{k=1}^{n}|A(\lambda_k)|^2\right) = \exp\left(-\frac{3}{16n}\sum_{k=1}^{n}|A_1+A_2+A_3|^2\right) \tag{6.21}$$

From the proof of Theorem 1 and Assumption (G.1), the first term in (6.21) is bounded by

$$
\begin{aligned}
-\frac{1}{16n}\sum_{k=1}^{n}\left(A_1\overline{A_1}\right) &\simeq -\frac{3}{16}\frac{1}{8\pi^2}\sum_{t=[\tau n]+1}^{[\tau n+\rho]}\sum_{s=[\tau n]+1}^{[\tau n+\rho]}(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\Gamma(t-s)\boldsymbol{z}_s(\boldsymbol{\beta}-\boldsymbol{\alpha}) \\
&\leq -\frac{3}{16}\frac{1}{8\pi^2}\sum_{t=[\tau n]+1}^{[\tau n+\rho]}\{(\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_t\}^2\times\min_{\lambda} f(\lambda)^{-1} \\
&= -\left[O(\rho)\right] \qquad (6.22)
\end{aligned}
$$

for $\rho > 0$. We have already shown in (6.16) and (6.17) that

$$
\frac{1}{16n}\sum_{k=1}^{n}\left\{A_1\overline{(A_2+A_3)}\right\} = O(n^{-1/2}) \text{ and } \frac{1}{16n}\sum_{k=1}^{n}\left\{\overline{A_1}(A_2+A_3)\right\} = O(n^{-1/2}). \quad (6.23)
$$

Furthermore, from the proof of Theorem 1 we can find a positive definite matrix $\boldsymbol{K}$ so that

$$
\frac{3}{16n}\sum_{k=1}^{n}(A_2+A_3)\overline{(A_2+A_3)} \simeq (\boldsymbol{a}',\boldsymbol{b}')\boldsymbol{K}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} \quad (6.24)
$$

Hence (6.22)-(6.24) implies the required result.

Proof of Lemma 2. Let $\boldsymbol{\theta}_1' = (\boldsymbol{\alpha}_1', \boldsymbol{\beta}_1', \tau_1)'$ and $\boldsymbol{\theta}_2' = (\boldsymbol{\alpha}_2', \boldsymbol{\beta}_2'\tau_2)'$ are some given values in $\boldsymbol{\Theta}$, and are the forms of $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha} + n^{-1/2}\boldsymbol{a}_1, \boldsymbol{\beta}_1 = \boldsymbol{\beta} + n^{-1/2}\boldsymbol{b}_1, \tau_1 = \tau + n^{-1}\rho_1, \boldsymbol{\alpha}_2 = \boldsymbol{\alpha} + n^{-1/2}\boldsymbol{a}_2, \boldsymbol{\beta}_2 = \boldsymbol{\beta} + n^{-1/2}\boldsymbol{b}_1$ and $\tau_2 = \tau + n^{-1}\rho_2$. Denoting $A(\lambda_k)$ under $\boldsymbol{\theta}_i$ as $A(\boldsymbol{a}_i, \boldsymbol{b}_i, \rho_i; \lambda_k)$ we set

$$
\begin{aligned}
\Delta_{1n} &= A(\boldsymbol{a}_1,\boldsymbol{b}_1,\rho_1;\lambda_k) - A(\boldsymbol{a}_2,\boldsymbol{b}_2,\rho_2;\lambda_k) \\
\Delta_{2n} &= |A(\boldsymbol{a}_1,\boldsymbol{b}_1,\rho_1;\lambda_k)|^2 - |A(\boldsymbol{a}_2,\boldsymbol{b}_2,\rho_2;\lambda_k)|^2
\end{aligned}
$$

and

$$
Y_n = \exp\left[-\frac{1}{8\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\{d_n(\lambda_k)\Delta_{1n} + \overline{d_n(\lambda_k)}\ \overline{\Delta_{1n}}\} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right].
$$

The process Y_n is written as

$$
Y_n = \left(\frac{L_n(\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2,\tau_2)}{L_n(\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\tau_1)}\right)^{1/4}. \quad (6.25)
$$

Then we observe

$$
\begin{aligned}
& E_{\alpha,\beta,\tau}\left|Z_n^{1/4}(\boldsymbol{a}_1,\boldsymbol{b}_1,\rho_1) - Z_n^{1/4}(\boldsymbol{a}_2,\boldsymbol{b}_2,\rho_2)\right|^{1/4} \\
=\ & E_{\alpha_1,\beta_1,\tau_1}\left(1-Y_n\right)^4 \\
=\ & E\left(1-4Y_n+6Y_n^2-4Y_n^3+Y_n^4\right)
\end{aligned}
$$

We have

$$
\begin{aligned}
-4EY_n &= -4E\exp\left(-\frac{1}{8\sqrt{n}}\sum_{k=1}^{n} f(\lambda_k)^{-1/2}\{d_n(\lambda_k)\Delta_{1n} + \overline{d_n(\lambda_k)}\ \overline{\Delta_{1n}}\} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right) \\
&= -4\int\cdots\int C_1 \exp\left[-\sum_{k=1}^{n}\left\{\frac{d_n(\lambda_k)}{f_k^{1/2}} + \frac{\overline{\Delta_{1n}}}{8\sqrt{n}}\right\}\left\{\frac{\overline{d_n(\lambda_k)}}{f_k^{1/2}} + \frac{\Delta_{1n}}{8\sqrt{n}}\right\}\right] \\
&\quad \times \exp\left[\frac{1}{64n}\sum_{k=1}^{n}\Delta_{1n}\overline{\Delta_{1n}} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right]\boldsymbol{d}(d_n(\lambda_1)\cdots d_n(\lambda_n)) \\
&= -4\exp\left[\frac{1}{64n}\sum_{k=1}^{n}\Delta_{1n}\overline{\Delta_{1n}} - \frac{1}{8n}\sum_{k=1}^{n}\Delta_{2n}\right] = -4\exp(\eta+\gamma) \quad \text{say}
\end{aligned}
$$

Similarly, we obtain

$$
6EY_n^2 = 6\exp(4\eta+2\gamma), \qquad -4EY_n^3 = -4\exp(9\eta+3\gamma)
$$

and

$$
EY_n^4 = \exp(16\eta+4\gamma).
$$

Hence

$$
E[1-Y_n]^4 = 1 - 4e^{\eta+\gamma} + 6e^{4\eta+2\gamma} - 4e^{9\eta+3\gamma} + e^{16\eta+4\eta}. \tag{6.26}
$$

Using the following expansion for small y

$$
e^y \simeq 1 + y
$$

we have

$$
\begin{aligned}
E[1-Y_n]^4 &= 1 - 4(1+\eta+\gamma) + 6(1+4\eta+2\gamma) - 4(1+9\eta+3\gamma) + (1+16\eta+4\gamma) \\
&\quad + O(\eta^2) + O(\gamma^2) + O(\eta\gamma) \\
&= 0 + O(\eta^2) + O(\gamma^2) + O(\eta\gamma)
\end{aligned}
$$

which implies that the Taylor expansion of (6.26) starts with the linear combinations of second order terms of η^2, γ^2 and $\eta\gamma$. Here we need to evaluate the asymptotics of η and γ in (6.26). Assume that without loss of generality $\rho_1 \geq \rho_2$, then

$$
\begin{aligned}
\Delta_{1n} &= \frac{1}{\sqrt{2\pi}} f(\lambda_k)^{-1/2} \sum_{s=[\tau n+\rho_2]+1}^{[\tau n+\rho_1]} (\boldsymbol{\beta}-\boldsymbol{\alpha})'\boldsymbol{z}_s e^{-is\lambda_k} \\
&\quad - \frac{1}{\sqrt{2\pi n}} f(\lambda_k)^{-1/2}\left(\sum_{s=1}^{[\tau n+\rho_1]} (\boldsymbol{a}_1-\boldsymbol{a}_2)'\boldsymbol{z}_s e^{-is\lambda_k} + \sum_{s=[\tau n+\rho_1]+1}^{n} (\boldsymbol{b}_1-\boldsymbol{a}_2)'\boldsymbol{z}_s e^{-is\lambda_k}\right).
\end{aligned}
$$

Using the similar argument in proof of Lemma 1, we observe

$$\eta = O\left[(\rho_1 - \rho_2)\right] + O\left[((\boldsymbol{a}_1 - \boldsymbol{a}_2)', (\boldsymbol{b}_1 - \boldsymbol{b}_2)')\boldsymbol{K}\begin{pmatrix} \boldsymbol{a}_1 - \boldsymbol{a}_2 \\ \boldsymbol{b}_1 - \boldsymbol{b}_2 \end{pmatrix}\right],$$

which is written as

$$\eta = O[(\rho_2 - \rho_1)] + O(\|\boldsymbol{a}_1 - \boldsymbol{a}_2\|) + O(\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|).$$

Analogously we have

$$\gamma = O[(\rho_2 - \rho_1)] + O(\|\boldsymbol{a}_1 - \boldsymbol{a}_2\|) + O(\|\boldsymbol{b}_1 - \boldsymbol{b}_2\|),$$

which completes the proof.

Proof of Theorem 2. The proof follows from Theorem 1, Lemmas 1 and 2 of this paper and Theorem 1.10.1 of Ibragimov and Has'minski (1981).

Proof of Theorem 3. The properties of the likelihood ratio $Z_n(\boldsymbol{a}, \boldsymbol{b}, \rho)$ established in Theorem 1, Lemmas 1 and 2 allow us to refer to Theorem 1.10.2 of Ibragimov and Has'minski (1981).

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