Detection of Structural Change in the Long-Run Persistence in a Univariate Time Series

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1. Introduction

In this paper, we investigate the test for structural change in the long-run persistence in a univariate time series. Our model has a unit root with no structural change under the null hypothesis, while under the alternative it changes from a unit-root process to a stationary one or vice versa. We propose the Lagrange Multiplier (LM) test and the 'demeaned version' of the LM test, and investigate both the asymptotic and finite-sample properties. Although the LM test is preferred in view of its limiting power, we recommend using the demeaned version of the test because it performs better in finite samples.

2. Testing for stability in the long-run persistence

2.1. Tests with a known break point

Let us consider the following model:

$$y_t = \mu_0 + \mu_1 t + x_t, \quad (1 - \alpha_t L)\psi(L)x_t = u_t,$$
(1)

for $t = 1, \dots, T$, where $\{u_t\} \sim i.i.d.(0, \sigma^2)$, L denotes a lag operator, $\psi(L)$ is the p-th order lag polynomial and all roots of $\psi(z) = 0$ lie outside the unit circle. Suppose that some shock occurred at time T_B^* and $T_B^*/T = \lambda^*$ is constant.

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The testing problem is that

$$H_0: \ \alpha_t = 1 \quad \forall t \ \text{v.s.} \ H_1^1: \begin{cases} \alpha_t = 1 & t \le T_B^* \\ |\alpha_t| < 1 & t \ge T_B^* + 1 \end{cases} \text{ or } H_1^2: \begin{cases} |\alpha_t| < 1 & t \le T_B^* \\ \alpha_t = 1 & t \ge T_B^* + 1 \end{cases}.$$
(2)

Note that $\{x_t\}$ is a unit-root process under H_0 . On the other hand, under H_1^1 , it changes from a unit-root process to a stationary one, while the change is in the reverse direction under H_1^2 .

The process $\{x_t\}$ in (1) can be expressed as

$$\Delta x_t = \rho_t x_{t-1} + \phi_{1t} \Delta x_{t-1} + \dots + \phi_{pt} \Delta x_{t-p} + u_t, \tag{3}$$

where

$$\rho_t = -(1 - \alpha_t)\psi(1),$$

$$\phi_{jt} = \alpha_t \psi_j - (1 - \alpha_t)(\psi_{j+1} + \dots + \psi_p), \quad 1 \le j \le p - 1, \quad \phi_{pt} = \alpha_t \psi_p.$$

Then, the testing problem (2) is equivalent to

$$H'_{0}: \ \rho_{t} = 0 \quad \forall t \ \text{v.s.} \ H^{1\prime}_{1}: \begin{cases} \rho_{t} = \rho_{1} = 0 & t \leq T^{*}_{B} \\ \rho_{t} = \rho_{2} < 0 & t \geq T^{*}_{B} + 1 \end{cases} \text{ or } H^{2\prime}_{1}: \begin{cases} \rho_{t} = \rho_{1} < 0 & t \leq T^{*}_{B} \\ \rho_{t} = \rho_{2} = 0 & t \geq T^{*}_{B} + 1 \end{cases}$$

$$(4)$$

Let us consider the LM test. After some algebra, the LM test statistic is shown to be

$$LM_{o}^{\tau}(\lambda^{*}) = \frac{\left(\sum_{t=1}^{T_{B}^{*}} \tilde{u}_{t}\tilde{x}_{t-1}\right)^{2}}{\tilde{\sigma}^{2}\sum_{t=1}^{T_{B}^{*}} \tilde{x}_{t-1}^{2}} + \frac{\left(\sum_{t=T_{B}^{*}+1}^{T} \tilde{u}_{t}\tilde{x}_{t-1}\right)^{2}}{\tilde{\sigma}^{2}\sum_{t=T_{B}^{*}+1}^{T} \tilde{x}_{t-1}^{2}}.$$
(5)

Note that the above test statistic (5) is constructed for the two-sided alternative. Since our testing problem (4) is one-sided, we modify the test statistic (5) as

$$LM_{1}^{\tau}(\lambda^{*}) = \frac{\sum_{t=1}^{T_{B}^{*}} \tilde{u}_{t}\tilde{x}_{t-1}}{\tilde{\sigma}\sqrt{\sum_{t=1}^{T_{B}^{*}} \tilde{x}_{t-1}^{2}}} + \frac{\sum_{t=T_{B}^{*}+1}^{T} \tilde{u}_{t}\tilde{x}_{t-1}}{\tilde{\sigma}\sqrt{\sum_{t=T_{B}^{*}+1}^{T} \tilde{x}_{t-1}^{2}}},$$

which rejects the null hypothesis when it takes small values.

As in Oya and Toda (1998), we can show that the above test statistic is asymptotically equivalent to the sum of the t statistics for ρ_1 and ρ_2 in the regression

$$\Delta \tilde{x}_t = \rho_1 D_{1t} \tilde{x}_{t-1} + \rho_2 D_{2t} \tilde{x}_{t-1} + \phi' \tilde{z}_{t-1} + e_t, \tag{6}$$

where $D_{1t} = 1$ for $t \leq T_B^*$ and 0 otherwise, $D_{2t} = 1 - D_{1t}$, $\phi = [\phi_1, \dots, \phi_p]'$ and $\tilde{z}_{t-1} = [\Delta \tilde{x}_{t-1}, \dots, \Delta \tilde{x}_{t-p}]'$. Then, we define the test statistic for (4) as

$$LM^{\tau}(\lambda^*) = t_1^{\tau}(\lambda^*) + t_2^{\tau}(\lambda^*),$$

where t_1^{τ} and t_2^{τ} are t statistics for ρ_1 and ρ_2 .

We also consider the 'demeaned version' of $LM^{\tau}(\lambda^*)$, i.e., the sum of the t statistics for ρ_1 and ρ_2 in the regression

$$\Delta \tilde{x}_t = c_1 D_{1t} + \rho_1 D_{1t} \tilde{x}_{t-1} + c_2 D_{2t} + \rho_2 D_{2t} \tilde{x}_{t-1} + \phi' \tilde{z}_{t-1} + e_t$$

We denote the demeaned version statistic as $LM_d^{\tau}(\lambda^*)$.

The following theorem gives the limiting distributions of LM^{τ} and LM_d^{τ} under the null hypothesis. We define the following functionals of a stochastic process V(r) in generic forms,

$$S(\lambda^*) = \frac{\frac{1}{2}(V^2(\lambda^*) - \lambda^*)}{\sqrt{\int_0^{\lambda^*} V^2(s)ds}} + \frac{\frac{1}{2}(V^2(1) - V^2(\lambda^*) - (1 - \lambda^*))}{\sqrt{\int_{\lambda^*}^{1} V^2(s)ds}},$$

$$\equiv S_1(\lambda^*) + S_2(\lambda^*).$$

$$S_{d}(\lambda^{*}) = \frac{\frac{\lambda^{*}}{2} \left(V^{2}(\lambda^{*}) - \lambda^{*} \right) - V(\lambda^{*}) \int_{0}^{\lambda^{*}} V(s) ds}{\sqrt{\lambda^{*2} \int_{0}^{\lambda^{*}} V^{2}(s) ds - \lambda^{*} \left(\int_{0}^{\lambda^{*}} V(s) ds \right)^{2}}} + \frac{\frac{1 - \lambda^{*}}{2} \left(V^{2}(1) - V^{2}(\lambda^{*}) - (1 - \lambda^{*}) \right) - \left(V(1) - V(\lambda^{*}) \right) \int_{\lambda^{*}}^{1} V(s) ds}{\sqrt{(1 - \lambda^{*})^{2} \int_{\lambda^{*}}^{1} V^{2}(s) ds - (1 - \lambda^{*}) \left(\int_{\lambda^{*}}^{1} V(s) ds \right)^{2}}} = S_{d1}(\lambda^{*}) + S_{d2}(\lambda^{*}).$$
(7)

Theorem 1 Under H'_0 , $LM^{\tau}(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $LM^{\tau}_d(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$, where \xrightarrow{d} signifies convergence in distribution and V(r) is a standard Brownian bridge, V(r) = W(r) - rW(1), with W(r) a standard Brownian motion.

When y_t has no trend, that is, if we know $\mu_1 = 0$ in the model (1), we define $\tilde{x}_t = y_t - y_0$ and construct the test statistics $LM^{\mu}(\lambda^*)$ and $LM^{\mu}_d(\lambda^*)$ completely in the same way as $LM^{\tau}(\lambda^*)$ and $LM^{\tau}_d(\lambda^*)$.

Theorem 2 Under H'_0 , $LM^{\mu}(\lambda^*) \xrightarrow{d} S(\lambda^*)$ and $LM^{\mu}_d(\lambda^*) \xrightarrow{d} S_d(\lambda^*)$, where V(r) = W(r).

Critical points of the above limiting distributions are tabulated in Table 1a.

Next, we investigate the power properties of the test statistics. To simplify the investigation, we consider the simple model with p = 0, that is, we consider the model (1) with $(1 - \alpha_t L)x_t = u_t$. In the following, we do not consider the fixed alternative H_1^1 or H_1^2 but a sequence of local alternatives:

$$H_1^{1\ell}: \begin{cases} \alpha_t = \alpha_1 = 1 & t \le T_B^* \\ \alpha_t = \alpha_2 = 1 - \frac{\theta}{T} & t \ge T_B^* + 1 \end{cases} \text{ or } H_1^{2\ell}: \begin{cases} \alpha_t = \alpha_1 = 1 - \frac{\theta}{T} & t \le T_B^* \\ \alpha_t = \alpha_2 = 1 & t \ge T_B^* + 1 \end{cases},$$

where $\theta > 0$.

The limiting distributions of test statistics under the local alternatives can be derived in the same way as Theorems 1 and 2. Using such distributions, we can depict the local limiting power as a function of θ . Figure 1 shows the case when the model does not have a linear trend and $\lambda^* = 0.5$. We can see that both test statistics are more powerful against H_1^1 than H_1^2 , and LM^{τ} has a better power property than the demeaned version statistic. On the other hand, from Figure 2, where a linear trend is included, LM_d^{τ} is more powerful than LM^{τ} when the alternative is H_1^1 .

2.2. Tests with an unknown break point

In practice, it is often the case that we do not know the actual break point T_B^* and, for such a case, several testing procedures have been proposed in the literature. One useful method is to take infimum of the test statistic in the closed interval:

$$\inf -LM^{\tau} = \inf_{\lambda \in \Lambda} LM^{\tau}(\lambda),$$

where Λ is a closed set in (0, 1). We also consider the test statistics of an average exponential form, as considered in Andrews, Lee and Ploberger (1996) and Andrews and Ploberger (1994),

$$avg-LM^{\tau} = \int_{\lambda \in \Lambda} \left(t_1^{\tau}(\lambda)^2 + t_2^{\tau}(\lambda)^2 \right) d\lambda, \quad \overline{exp}-LM^{\tau} = \log \int_{\lambda \in \Lambda} \exp\left(t_1^{\tau}(\lambda)^2 + t_2^{\tau}(\lambda)^2 \right) d\lambda.$$

Exactly in the same way, we consider the test statistics $inf-LM_d^{\tau}$, $avg-LM_d^{\tau}$ and $\overline{exp}-LM_d^{\tau}$ as the demeaned versions.

The limiting distributions are derived in the same way as the previous subsection. We tabulate the percentage points of the limiting distributions when $\Lambda = [0.2, 0.8]$ in Table 1b. Notice that *inf-LM*-type test rejects the null hypothesis when it takes small values while $avg-LM \ \overline{exp}-LM$ -type tests reject it for large values.

3. Finite-sample properties

In this section, we investigate the finite-sample properties of the test statistics in the previous section. The following data generating process is considered:

$$y_t = x'_t \beta + z_t, \quad (1 - \alpha_t L)(1 - \psi L)z_t = u_t,$$

where $\{u_t\}$ is NID(0,1) and $x_t = 1$ or [1,t]. We set $\beta = 0$, $\alpha_1, \alpha_2 = 1$, 0.95, 0.9, 0.8, 0.7, $\psi = 0, \pm 0.5, \lambda^* = 0.3, 0.5, 0.7$, and the sample size T = 100 and 200. The initial value of z_t is set equal to 0 and the first 100 samples are deleted. The level of significance is 0.05 and the number of replications is 1,000 in all experiments, performed by the GAUSS matrix programming language.

Tables 2a and 2b report the size and power without a linear trend when the break point is known and $\psi = 0$. From the tables, we can see that LM^{μ} has a reasonable empirical size close to 0.05 in all cases, while LM_d^{τ} tends to overly reject the null hypothesis when T = 100. As to the power, in almost all the cases, LM_d^{μ} is more powerful than LM^{μ} . Since the power of LM^{μ} is very low, we do not recommend the use of LM^{μ} in practice. As we have seen in the previous section, the demeaned version test seems to be more powerful against H_1^1 for smaller values of λ^* and so against H_1^2 for larger values of λ^* . From Tables 3a and 3b, we see that the relative performance of the test is preserved when a linear trend is included.

For other values of ψ , the performance of the test is similar to the case of $\psi = 0$, but the tests tend to be slightly less powerful when $\psi = 0.5$. (We omit details to save space.)

As we have seen in the above, our finite-sample simulation shows that the power of LM^{μ} is very low, although LM^{μ} performs better than LM_d^{μ} , in view of the power from the asymptotic result. This poor performance under the alternative is due to the initial-value condition. Table 4 summarizes the effect of the initial value on the power. We see that

 LM^{μ} has reasonable power when $z_0 = 0$, whereas its power decreases dramatically when $x_0 = 10$, even if the sample size is 1,000. On the other hand, LM_d^{μ} seems to be robust to the initial-value condition. From these results, we recommend using the demeaned test statistic in practical analyses.

For the case when the break point is unknown, we tabulate the simulation results in Tables 5–6. We only report the demeaned versions of the test statistics and the case when $\lambda^* = 0.5$ because the performance of the LM-type test is very poor. The sizes of the three test statistics are similar but the *avg-LM* statistic has a little larger size. As to the power, the \overline{exp} -LM statistic is the most powerful against H_1^1 among three statistics while the power of *avg-LM* statistic is highest against H_1^2 . On average, it seems that \overline{exp} -LM test has totally the best finite-sample properties among the three test statistics.

4. Concluding remarks

In this paper, we have investigated the test for a change in the long-run persistence in a univariate time series. We proposed two types of test statistics, one is the LM-type test statistic and the other is the 'demeaned version' of the LM-type test. Although the former performs better than the latter, in view of the limiting power property in some cases, we recommend using the demeaned version of the LM-type test because the finite-sample property of the LM-type test is much affected by the initial condition and it loses power in some cases, while the demeaned version is robust to such a condition.

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