Uniform and L_1-Norm Error Bounds in Asymptotic Expansions of Multivariate Scale Mixtures

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Abstract

This paper deals with the distribution of multivariate scale mixture variate defined by $\boldsymbol{X} = S\boldsymbol{Z}$, where $\boldsymbol{Z} = (Z_1, \cdots, Z_p)'$, $Z_1, \ldots, Z_p$ are *i.i.d.* random variables, and S is a positive definite random matrix independent of $\boldsymbol{Z}$. First we obtain asymptotic expansions of the distribution function and the density function of $\boldsymbol{X}$ when $S = \mathrm{diag}(S_1, \ldots, S_p)$. Uniform error bounds are given for approximations of the distribution function of $\boldsymbol{X}$. L_1-norm error bounds are given for approximations for the density function of $\boldsymbol{X}$. Then it is shown how our results can be extended for the general case when the scale matrix may be not necessary diagonal. The L_1-norm error bounds are applied in obtaining error bounds for asymptotic expansions of Lawley-Hotelling's T_0^2 statistic.

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1. Introduction

Let $\boldsymbol{Z} = (Z_1, \cdots, Z_p)'$ be a random vector, where $Z_1, \ldots, Z_p$ are *i.i.d.* random variables, and G and g be the distribution function and the density functions of Z_1, respectively. Further, let S be a positive definite random matrix independent of $\boldsymbol{Z}$. Then we consider the distribution of

$$\boldsymbol{X} = S\boldsymbol{Z}, \tag{1.1}$$

which is called a multivariate scale mixture of $\boldsymbol{Z}$. Here it is tacitly assumed that the scale factor S is close to I in some sense. In practical applications we consider two cases when Z_1 has the standard normal distribution or a gamma distribution. Asymptotic expansions and their error bounds in the univariate case have been extensively studied. For a summary, see, e.g., Fujikoshi and Shimizu (1990), Fujikoshi (1993), Shimizu and Fujikoshi (1997), Ulyanov, Fujikoshi and Shimizu (1999). Having in mind statistical applications of our results we consider a transformation given by

$$S = Y^{\delta\rho} \quad \text{or} \quad Y = S^{\delta/\rho}, \tag{1.2}$$

where $\delta = \pm 1$ and $1/\rho$ is a positive integer. For multivariate scale mixtures, some special cases have been studied. For the distribution function, Fujikoshi and Shimizu (1989a) treated the case $S = sI_p$. Fujikoshi and Shimizu (1989b) treated the case $S - I \geq 0, G = \Phi, \delta = 1$ and $\alpha = 1/2$, where Φ is the distribution function of $N(0, 1)$. For the density function, Shimizu (1995) obtained L_1-error bound when $G = \Phi, \delta = 1$ and $\alpha = 1/2$.

First we consider the case when $S = \mathrm{diag}(S_1, \ldots, S_p)$. Then the transformation (1.2) is expressed as

$$Y_j = S_j^{\delta/\rho}, \quad j = 1, \ldots, p, \tag{1.3}$$

where $Y = \mathrm{diag}(Y_1, \ldots, Y_p)$. Uniform error bounds are given for approximations of the distribution function of $\boldsymbol{X}$. L_1-norm error bounds are given for approximations for the density function of $\boldsymbol{X}$. From the latter results we can obtain asymptotic expansions for $\mathrm{P}(\boldsymbol{X} \in A)$ and their error bounds for any Borel set A. It is shown how our results can be extended for the general case when the scale matrix may be not necessary diagonal. The L_1-norm error bounds are applied in obtaining error bounds for asymptotic expansions of Lawley- Hotelling's T^2 statistic.

2. Uniform Error Bounds

Let G and g be the distribution function and the density function of Z_1, respectively. Let $D = \{x \in \mathbf{R} : g(x) > 0\}$. Assume that

A1. G is k times continuously differentiable on D.

The distribution function of $X = SZ$ given $S = s = y^{\delta\rho}$ is given by $G(xy^{-\delta\rho})$. For $j = 1, \dots, k$ and for $x \in D$, let $c_{\delta,j}(x)$ be defined by

$$\frac{\partial^j}{\partial y^j} G(xy^{-\delta\rho}) = y^{-j} c_{\delta,j}(xy^{-\delta\rho}) g(xy^{-\delta\rho}), \tag{2.1}$$

and $c_{\delta,j}(x) = 0$ for $x \notin D$ and write

$$\alpha_{\delta,j} \equiv \begin{cases} 1, & \text{if } j = 0, \\ (1/j!) \sup_x |c_{\delta,j}(x)| g(x), & \text{if } j \geq 1. \end{cases}$$

Note that if $p = 1$ we can take

$$\alpha_{\delta,0} = \min\{G(0), 1 - G(0)\}.$$

However, $\alpha_{\delta,0} = 1$ for all $p \geq 2$. The functions $c_{\delta,j}(x)$ may be defined also by

$$\left. \frac{\partial^j}{\partial y^j} G(xy^{-\delta\rho}) \right|_{y=1} = c_{\delta,j}(x) g(x).$$

For explicit expressions of $c_{\delta,j}(x)$ in normal or Gamma distributions, see, e.g. Fujikoshi and Shimizu (1990), Fujikoshi (1993), etc. The distribution function of $\boldsymbol{X} = (X_1, \dots, X_p)$ in (1.1) with $S = \mathrm{diag}(S_1, \dots, S_p)$ can be written as

$$\begin{aligned} F_p(\boldsymbol{x}) &= \mathrm{P}(X_1 \leq x_1, \dots, X_p \leq x_p) \\ &= \mathrm{E}[G(x_1 Y_1^{-\delta\rho}) \dots G(x_p Y_p^{-\delta\rho})], \end{aligned}$$

where $\boldsymbol{x} = (x_1, \dots, x_p)$. Let $G_p(\boldsymbol{x}) = G(x_1) \dots G(x_p)$. We consider an approximation for $F_p(\boldsymbol{x})$,

$$G_{\delta,k,p}(\boldsymbol{x}) = \mathrm{E}\left[G_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{j!} \left((Y_1 - 1)\frac{\partial}{\partial y_1} + \dots + (Y_p - 1)\frac{\partial}{\partial y_p} \right)^j \right.$$

$$
\begin{aligned}
& \quad \times G(x_1 y_1^{-\delta\rho}) \dots G(x_p y_p^{-\delta\rho}) \Big|_{y_1=\dots=y_p=1} \Bigg] \\
&= \; G_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \sum_{(j)} \frac{1}{j_1! \dots j_p!} c_{\delta,j_1}(x_1) \dots c_{\delta,j_p}(x_p) g_p(\boldsymbol{x}) \qquad (2.2) \\
& \quad \times \mathrm{E}\left[(Y_1 - 1)^{j_1} \dots (Y_p - 1)^{j_p} \right],
\end{aligned}
$$

where $g_p(\boldsymbol{x}) = g(x_1) \dots g(x_p)$ and the sum $\sum_{(j)}$ is taken over all non-negative integers such that $j_1 + \dots + j_p = j$.

Let

$$
W_{\delta,j,p} = \sum_{[j]} \frac{(p-1)!}{i_1! \dots i_m!} \alpha_{\delta,j_1} \dots \alpha_{\delta,j_p}, \qquad (2.3)
$$

where summation is taken over all non-negative integers $0 \le j_1 \le \dots \le j_p$ such that $j_1 + \dots + j_p = j$, and the constants $m, i_1, \dots, i_m$ are positive integers such that

$$
\begin{aligned}
& 0 \le j_1 = \dots = j_{i_1} < j_{i_1+1} = \dots = j_{i_1+i_2} \\
& < \dots < j_{i_1+\dots+i_{m-1}+1} = \dots = j_{i_1+\dots+i_m} (= j_p) \le j.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
W_{\delta,1,p} &= \alpha_{\delta,1}, \\
W_{\delta,2,p} &= \alpha_{\delta,2} + \frac{1}{2}(p-1)\alpha_{\delta,1}^2, \\
W_{\delta,3,p} &= \alpha_{\delta,3} + (p-1)\alpha_{\delta,1}\alpha_{\delta,2} + \frac{1}{6}(p-1)(p-2)\alpha_{\delta,1}^3, \qquad (2.4) \\
W_{\delta,4,p} &= \alpha_{\delta,4} + \frac{1}{2}\alpha_{\delta,2}^2 + (p-1)\alpha_{\delta,1}\alpha_{\delta,3} \\
& \quad + \frac{1}{2}(p-1)(p-2)\alpha_{\delta,1}^2\alpha_{\delta,2} + \frac{1}{24}(p-1)(p-2)(p-3)\alpha_{\delta,1}^4.
\end{aligned}
$$

Theorem 2.1. *Let $\boldsymbol{X} = S\boldsymbol{Z}$ be a multivariate scale mixture in* (1.1) *with $S = \mathrm{diag}(S_1, \dots, S_p)$. Suppose that the distribution G of Z_1 satisfies* A1 *and* $\mathrm{E}(Y_i^k) < \infty, i = 1, \dots, p$ *for a given integer k. Then we have*

$$
|F_p(\boldsymbol{x}) - G_{\delta,k,p}(\boldsymbol{x})| \le \beta_{\delta,k,p} \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k], \qquad (2.5)
$$

where $\beta_{\delta,1,p} = 1 + W_{\delta,1,p}$ *and for* $k \geq 2$

$$\beta_{\delta,k,p} = \left\{ W_{\delta,k,p}^{1/k} + \left(1 + p\sum_{j=1}^{k-1} W_{\delta,j,p}\right)^{1/k} \right\}^k . \tag{2.6}$$

Theorem 2.2. *Suppose that the conditions of Theorem 2.1 are satisfied. Then we have*

$$|F_p(\boldsymbol{x}) - G_{\delta,k,p}(\boldsymbol{x})| \leq \gamma_{\delta,k,p} \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k], \tag{2.7}$$

where $\gamma_{\delta,k,p}$ *is defined by formula for* $k \geq 2$

$$\gamma_{\delta,k,p} = p^{-1}\left(\beta_{\delta,k} + (p-1)\sum_{q=0}^{k-1} \gamma_{\delta,k-q,p-1}\alpha_{\delta,q}\right) \tag{2.8}$$

with $\gamma_{\delta,1,p} = \beta_{\delta,1}, \gamma_{\delta,k,0} = 0$ *and* $\gamma_{\delta,k,1} = \beta_{\delta,k}$ *for all* $k \geq 1$*; here*

$$\beta_{\delta,k} = \left(\alpha_{\delta,k}^{1/k} + (\alpha_{\delta,0} + \ldots + \alpha_{\delta,k-1})^{1/k}\right)^k .$$

In particular, we have for all $p \geq 1$

$$\begin{aligned} \gamma_{\delta,1,p} &= \beta_{\delta,1}, \\ \gamma_{\delta,2,p} &= \beta_{\delta,2} + \frac{1}{2}(p-1)\alpha_{\delta,1}\beta_{\delta,1}. \end{aligned}$$

It is important to note that in the case $\delta = 1, \rho = 1/2$, and $Z_1 \sim N(0,1)$ we have the following remarkable property:

$$\left.\frac{\partial^j}{\partial y^j}\Phi(xy^{-1/2})\right|_{y=1} = -2^{-j}H_{2j-1}(x)\varphi(x), \tag{2.9}$$

where $H_n(x)$ is Chebyshev-Hermite polynomial of degree n defined by the equality

$$H_n(x) = (-1)^n\{\phi(x)\}^{-1}\frac{d^n}{dx^n}\phi(x). \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\frac{\partial^j}{\partial y^j}\Phi(xy^{-1/2})\bigg|_{y=1} = 2^{-j}\frac{d^{2j}}{dx^{2j}}\Phi(x).$$

Therefore from (2.2) we can write $G_{\delta,k,p}$ in the form

$$G_{1,k,p}(\boldsymbol{x}) = \mathrm{E}\left[\Phi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1}\frac{1}{2^j j!}\left(\partial'_{\boldsymbol{x}}(S-I)\partial_{\boldsymbol{x}}\right)^j \Phi_p(\boldsymbol{x})\right], \tag{2.11}$$

where $\Phi_p(\boldsymbol{x}) = \Phi(x_1)\ldots\Phi(x_p)$ and $\partial_{\boldsymbol{x}}$ is the differential operator

$$\partial_{\boldsymbol{x}} = (\partial/\partial x_1, \ldots, \partial/\partial x_p)'.$$

The approximation (2.11) was considered by Fujikoshi and Shimizu (1989b) in a special case when

$$S - I \geq 0,\ \ \delta = 1,\ \ \rho = \frac{1}{2},\ \ G = \Phi$$

and without assumption that S is a diagonal matrix.

3. L_1-Norm Error Bounds

In this section we give main results on asymptotic expansions of the density function of $\boldsymbol{X}$ and their error bounds. First we consider the univariate case, i.e.,

$$X = SZ. \tag{3.1}$$

Let $f(x)$ and $g(z)$ be the probability density functions of X and Z, respectively. Assume that

A2. g is k times continuously differentiable on D.

Considering the transformation $Y = S^{\delta/\rho}$, we have

$$f(x) = \mathrm{E}[Y^{-\delta\rho}g(xY^{-\delta\rho})]. \tag{3.2}$$

We define a function $b_{\delta,j}(x)$ for $j \geq 1$ and for $x \in D$, by formula

$$\frac{\partial^j}{\partial y^j}\left(y^{-\delta\rho}g(xy^{-\delta\rho})\right) = y^{-j}y^{-\delta\rho}b_{\delta,j}(xy^{-\delta\rho})g(xy^{-\delta\rho}) \tag{3.3}$$

and $c_{\delta,j}(x) = 0$ for $x \notin D$. For $j = 0$ by $b_{\delta,0}(x) = 1$. The equality (3.3) can be easily checked by mathematical induction. Defferentiating both sides of (2.1) with respect to x, it is easy to see that

$$\frac{d}{dx}\left(c_{\delta,j}(x)g(x)\right) = b_{\delta,j}(x)g(x). \tag{3.4}$$

We define also for $j \geq 0$

$$\xi_{\delta,j} = \frac{1}{j!}\left\|b_{\delta,j}(x)g(x)\right\|_1, \tag{3.5}$$

where for any integrable function $h(x)$,

$$\|h(x)\|_1 = \int_{-\infty}^{\infty} |h(x)|dx,$$

and

$$\eta_{\delta,k} = \left\{\xi_{\delta,k}^{1/k} + \left(2 + \sum_{j=1}^{k-1}\xi_{\delta,j}\right)^{1/k}\right\}^k.$$

Consider an approximation $g_{\delta,k}(x,y)$ for $y^{-\delta\rho}g(xy^{-\delta\rho})$ defined as follows: for $y > 0$ and $x \in \mathbf{R}^1$,

$$g_{\delta,k}(x,y) = g(x) + \sum_{j=1}^{k-1}\frac{1}{j!}b_{\delta,j}(x)g(x)(y-1)^j,$$

where $g_{\delta,1}(x,y) = g(x)$. This suggests an approximation $g_{\delta,k}(x)$ for $f(x)$:

$$g_{\delta,k}(x) = \mathrm{E}[g_{\delta,k}(x,Y)]. \tag{3.6}$$

Note that

$$\frac{d}{dx}G_{\delta,k}(x) = g_{\delta,k}(x),$$

which is easily seen from (3.4).

Theorem 3.1. *Le X be a scale mixture of Z defined by (3.1). Suppose that the density function g of Z_1 satisfies* **A2** *and $\mathrm{E}(Y_i^k) < \infty, i = 1,\ldots,p$ for a given integer k. Then we have for any $k \geq 1$ and any Borel set $A \subset \mathbf{R}^1$*

$$|\mathrm{P}(X \in A) - \int_A g_{\delta,k}(x)dx| \leq \frac{1}{2}\eta_{\delta,k}\mathrm{E}[|Y-1|^k]. \tag{3.7}$$

Next we consider a p variate case. Let $f_p(\boldsymbol{x})$ and $g_p(\boldsymbol{z})$ be the density functions of $\boldsymbol{X}$ and $\boldsymbol{Z}$. Then $g_p(\boldsymbol{z}) = g(z_1)\dots g(z_p)$, and the conditional density of $\boldsymbol{X}$ given $Y_i = y_i, i = 1,\dots,p$ is given by

$$y_1^{-\delta\rho} g(xy_1^{-\delta\rho})\dots y_p^{-\delta\rho} g(xy_p^{-\delta\rho}).$$

We define an approximation for $f_p(\boldsymbol{x})$ as

$$\begin{aligned} g_{\delta,k,p}(\boldsymbol{x}) &= \mathrm{E}\Bigg[g_p(\boldsymbol{x}) + \sum_{j=1}^{k-1}\frac{1}{j!}\left((Y_1-1)\frac{\partial}{\partial y_1} + \dots + (Y_p-1)\frac{\partial}{\partial y_p}\right)^j \\ &\quad \times y_1^{-\delta\rho} g(x_1 y_1^{-\delta\rho})\dots y_p^{-\delta\rho} g(x_p y_p^{-\delta\rho})\Big|_{y_1=\dots=y_p=1}\Bigg] \\ &= g_p(\boldsymbol{x}) + \sum_{j=1}^{k-1}\sum_{(j)}\frac{1}{j_1!\dots j_p!} b_{\delta,j_1}(x_1)\dots b_{\delta,j_p}(x_p) g_p(\boldsymbol{x}) \\ &\quad \times \mathrm{E}\left[(Y_1-1)^{j_1}\dots(Y_p-1)^{j_p}\right], \end{aligned} \tag{3.8}$$

which is an extension of $g_{\delta,k}(\boldsymbol{x})$. Note that from (3.4) we have

$$g_{\delta,k,p}(\boldsymbol{x}) = \frac{\partial^p}{\partial x_1\dots\partial x_p} G_{\delta,k,p}(\boldsymbol{x}).$$

Put $\eta_{\delta,1,p} = 2 + V_{\delta,1,p}$ and for $k \ge 2$

$$\eta_{\delta,k,p} = \left\{ V_{\delta,k,p}^{1/k} + \left(2 + p\sum_{j=1}^{k-1} V_{\delta,j,p}\right)^{1/k}\right\}^k, \tag{3.9}$$

where

$$V_{\delta,j,p} = \sum_{[j]}\frac{(p-1)!}{i_1!\dots i_m!}\xi_{\delta,j_1}\dots\xi_{\delta,j_p}, \tag{3.10}$$

and the summation is taken in the sense of (2.3). Note that $V_{\delta,j,p}$ is expressed in the same form as the expression (2.4) for $W_{\delta,j,p}$.

Theorem 3.2. *Let $\boldsymbol{X} = S\boldsymbol{Z}$ be a multivariate scale mixture in* (1.1) *with $S = \mathrm{diag}(S_1,\dots,S_p)$. Suppose that the density function g of Z_1 satisfies* **A2** *and $\mathrm{E}(Y_i^k) < \infty, i = 1,\dots,p$ for a given integer k. Then we have for any Borel set $A \subset \mathbf{R}^p$*

$$|\mathrm{P}(\boldsymbol{X}\in A) - \int_A g_{\delta,k,p}(\boldsymbol{x})d\boldsymbol{x}| \le \frac{1}{2}\eta_{\delta,k,p}\sum_{i=1}^p \mathrm{E}[|Y_i - 1|^k]. \tag{3.11}$$

Theorem 3.3. *Under the same condition as in Theorem* 3.2 *we have for any Borel set* $A \subset \mathbf{R}^p$

$$|\mathrm{P}(\boldsymbol{X} \in A) - \int_A g_{\delta,k,p}(\boldsymbol{x}) d\boldsymbol{x}| \le \frac{1}{2} \nu_{\delta,k,p} \sum_{i=1}^{p} \mathrm{E}[|Y_i - 1|^k], \tag{3.12}$$

where $\nu_{\delta,k,p}$ *are determined recursively by the relation for* $k \ge 2$

$$\nu_{\delta,k,p} = p^{-1} \left(\eta_{\delta,k} + (p-1) \sum_{q=0}^{k-1} \nu_{\delta,k-q,p-1} \xi_{\delta,q} \right) \tag{3.13}$$

with $\nu_{\delta,1,p} = \eta_{\delta,1}, \nu_{\delta,k,0} = 0$ *and* $\nu_{\delta,k,1} = \eta_{\delta,k}$ *for all* $k \ge 1$.

Remark 3.1. In the case $\delta = 1$, $\rho = 1/2$ and $Z_1 \sim N(0,1)$ a similar result has been obtained in Theorem 2 in Shimizu (1995) with the same recurrence relation as (3.13) but with another value for $\nu_{\delta,k,1}$.

For the case $\delta = 1, \rho = 1/2$, and $Z_1 \sim N(0,1)$, we have

$$\frac{\partial^j}{\partial y^j} \left(y^{-1/2} \varphi(xy^{-1/2}) \right) \Big|_{y=1} = 2^{-j} H_{2j}(x) \varphi(x). \tag{3.14}$$

It follows from (2.9) and (3.14) that

$$\frac{\partial^j}{\partial y^j} \left(y^{-1/2} \varphi(xy^{-1/2}) \right) \Big|_{y=1} = 2^{-j} \frac{d^{2j}}{dx^{2j}} \varphi(x).$$

Therefore the approximations $g_{1,k,1}$ (see (3.8)) in this case can be written in the form (cf (2.10))

$$g_{1,k,1}(x) = \mathrm{E} \left[\varphi(x) + \sum_{j=1}^{k-1} \frac{1}{2^j j!} (Y_1 - 1)^j \frac{d^{2j}}{dx^{2j}} \varphi(x) \right]. \tag{3.15}$$

In the multivariate case $p > 1$ and $Y = \mathrm{diag}(Y_1, \ldots, Y_p)$ the function $g_{1,k,p}$ from (3.8) can be written in the form

$$g_{1,k,p}(\boldsymbol{x}) = \mathrm{E} \left[\varphi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{2^j j!} \left(\partial'_{\boldsymbol{x}} (Y - I) \partial_{\boldsymbol{x}} \right)^j \varphi_p(\boldsymbol{x}) \right], \tag{3.16}$$

where $\varphi_p(\boldsymbol{x}) = \varphi(x_1)\dots\varphi(x_p)$. As it was shown in Shimizu (1995) (see the proof of Theorem 2, p.135) the expression of $g_{1,k,p}$ in the form (3.16) enables us to extend Theorem 3.3 to the general case when the scale matrix may be not necessary diagonal.

In fact, assume that $\boldsymbol{X} = S\boldsymbol{Z}$, $\boldsymbol{Z}$ is distributed as the standard nomal distribution $N_p(0, I_p)$, and S is symmetric positive definite matrix and is not diagonal. Fix any Borel set $A \subset \mathbf{R}^p$. We have

$$\mathrm{P}(\boldsymbol{X} \in A) = \mathrm{E}_S[\mathrm{P}(\boldsymbol{X} \in A|S)], \tag{3.17}$$

where E_S denotes expectation with respect to S. It means we can construct at first approximation for $\mathrm{P}(\boldsymbol{X} \in A)$ for any given value of S and then taking expectation with respect to S we get result for $\mathrm{P}(\boldsymbol{X} \in A)$. In the following arguments S is non-random symmetric positive definite matrix. Under this assumption on S there exists orthogonal matrix T such that $S = TLT'$, where $L = \mathrm{diag}(L_1, \dots, L_p)$ and $L_i = Y_i^{1/2}$ for $i = 1, \dots, p$. We have

$$\mathrm{P}(S\boldsymbol{Z} \in A) = \mathrm{P}(LT'\boldsymbol{Z} \in T'A) = \mathrm{P}(L\boldsymbol{Z} \in T'A), \tag{3.18}$$

as $T'\boldsymbol{Z}$ has also the standard normal distribution in $\mathbf{R}^p$.

Note that

$$\int_{T'A} \varphi_p(\boldsymbol{x})d\boldsymbol{x} = \int_A \varphi_p(\boldsymbol{x})d\boldsymbol{x}, \tag{3.19}$$

as the standard normal distribution is invariant with respect to orthogonal transformations. Moreover, if we put $\boldsymbol{v} = T\boldsymbol{x}$, then $T\partial_{\boldsymbol{x}} = \partial_{\boldsymbol{v}}$ and therefore we have that $\partial'_{\boldsymbol{x}}(Y - I)\partial_{\boldsymbol{x}} = \partial'_{\boldsymbol{v}}T(Y-I)T'\partial_{\boldsymbol{v}} = \partial'_{\boldsymbol{v}}(S^2 - I)\partial_{\boldsymbol{v}}$. Thus, we get for any $j = 1, 2, \dots, k-1$

$$\int_{T'A} (\partial'_{\boldsymbol{x}}(Y - I)\partial_{\boldsymbol{x}})^j \varphi_p(\boldsymbol{x})d\boldsymbol{x} = \int_A \left(\partial'_{\boldsymbol{v}}(S^2 - I)\partial_{\boldsymbol{v}}\right)^j \varphi_p(\boldsymbol{v})d\boldsymbol{v}. \tag{3.20}$$

Combining (3.16)$\sim$ (3.20) we can rewrite (3.12) in the form

$$\begin{aligned} \Bigg|\mathrm{P}(\boldsymbol{X} \in A) \;\; &- \;\; \int_A \mathrm{E}\left[\varphi_p(\boldsymbol{x}) + \sum_{j=1}^{k-1} \frac{1}{2^j j!}\left(\partial'_{\boldsymbol{x}}(S^2 - I)\partial_{\boldsymbol{x}}\right)^j \varphi_p(\boldsymbol{x})\right] d\boldsymbol{x}\Bigg| \\ &\leq \;\; \frac{1}{2}\nu_{\delta,k,p}\mathrm{E}[\mathrm{tr}(S^2 - I)^k], \end{aligned} \tag{3.21}$$

provided $(S^2 - I)^k$ is positive definite matrix, as in this case

$$\mathrm{tr}(S^2 - I)^k = \sum_{k=1}^{p}(Y_i - 1)^k.$$

The similar extension is possible for Theorem 3.2 as well.

It is necessary to note that for $\delta = -1$ or $+1$ and for any positive ρ we have

$$\frac{\partial}{\partial y}\left(y^{-\delta\rho}\varphi(xy^{-\delta\rho})\right)\bigg|_{y=1} = \delta\rho H_2(x)\varphi(x). \tag{3.22}$$

Therefore the previous arguments imply the parts (i) and (ii) in the following.

Theorem 3.4. *Let $k = 1, 2$ or 3 and $\boldsymbol{X} = S\boldsymbol{Z}$ be a multivariate scale mixture with $\boldsymbol{Z} \sim N_p(0, I_p)$ and S is such that $(S^{\delta/\rho} - I)$ is positive definite matrix and $\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^k] < \infty$. Then for any Borel set $A \subset \mathbf{R}^p$ we have*
(i) *for $k = 1$:*

$$\left|\mathrm{P}(\boldsymbol{X} \in A) - \int_A \varphi_p(\boldsymbol{x})d\boldsymbol{x}\right| \leq \frac{1}{2}\min\{\eta_{\delta,1,p}, \nu_{\delta,1,p}\}\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)],$$

(ii) *for $k = 2$:*

$$\left|\mathrm{P}(\boldsymbol{X} \in A) - \int_A \mathrm{E}\left[\varphi_p(\boldsymbol{x}) + (\delta\rho)\left(\partial'_{\boldsymbol{x}}(S^{\delta/\rho} - I)\partial_{\boldsymbol{x}}\right)\varphi_p(\boldsymbol{x})\right]d\boldsymbol{x}\right|$$
$$\leq \frac{1}{2}\min\{\eta_{\delta,2,p}, \nu_{\delta,2,p}\}\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^2],$$

(iii) *for $k = 3$:*

$$\Bigg|\mathrm{P}(\boldsymbol{X} \in A) - \int_A \mathrm{E}\Bigg[\varphi_p(\boldsymbol{x}) + \sum_{j=1}^{2}\frac{(\delta\rho)^j}{j!}\left(\partial'_{\boldsymbol{x}}(S^{\delta/\rho} - I)\partial_{\boldsymbol{x}}\right)^j\varphi_p(\boldsymbol{x})$$
$$+ (\rho^2 - \frac{1}{2}\delta\rho)\left(\partial'_{\boldsymbol{x}}(S^{\delta/\rho} - I)^2\partial_{\boldsymbol{x}}\right)\varphi_p(\boldsymbol{x})\Bigg]d\boldsymbol{x}\Bigg|$$
$$\leq \frac{1}{2}\min\{\eta_{\delta,3,p}, \nu_{\delta,3,p}\}\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^3].$$

The part (ii) in Theorem 3.4 holds without assumption that $(S^{\delta/\rho} - I)$ is positive definite matrix. Moreover if $(S^{\delta/\rho} - I)$ is not positive definite, the in Theorm 3.4 (i), (iii) we can use inequalities

$$\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)] \leq p^{1/2}\left(\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^2]\right)^{1/2},$$

and

$$\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^3] \le \left(\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^4]\right)^{3/4},$$

provided that $\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^2] < \infty$ and $\mathrm{E}[\mathrm{tr}(S^{\delta/\rho} - I)^4] < \infty$, respectively. The inequlities follows from Holder'sinequality. In order to derive the result (iii), it is enough to show that

$$\begin{aligned} & \int_{T'A} \left(\sum_{i=1}^{p} (Y_i - 1) \frac{\partial}{\partial z_i} \right)^2 h_\varphi(\boldsymbol{x}, \boldsymbol{z}) \bigg|_{z_1 = \ldots = z_p = 1} d\boldsymbol{x} \\ = & \int_{T'A} \left[\rho^2 \left(\partial'_{\boldsymbol{x}} (Y - I) \partial_{\boldsymbol{x}} \right)^2 \right. \\ + & \left. (2\rho^2 - \delta\rho) \left(\partial'_{\boldsymbol{x}} (Y - I)^2 \partial_{\boldsymbol{x}} \right) \right] \varphi_p(\boldsymbol{x}) d\boldsymbol{x}, \end{aligned} \tag{3.23}$$

where $h_\varphi(\boldsymbol{x}, \boldsymbol{z}) = \prod_{i=1}^{p} z_i^{-\delta\rho} \varphi(x_i z_i^{-\delta\rho})$. We have

$$\frac{\partial^2}{\partial z^2} \left(z^{-\delta\rho} g(x z^{-\delta\rho}) \right) \bigg|_{z=1} = \left[\rho^2 x^4 - (4\rho^2 + \delta\rho) x^2 + \rho^2 + \delta\rho \right] \varphi(x). \tag{3.24}$$

Put

$$M = \left(\sum_{i=1}^{p} (Y_i - 1) \frac{\partial}{\partial z_i} \right)^2 h_\varphi(\boldsymbol{x}, \boldsymbol{z}) \bigg|_{z_1 = \ldots = z_p = 1}.$$

It follows from (3.22) and (3.24) that

$$\begin{aligned} M = & \left[\sum_{i=1}^{p} (y_i - 1)^2 \{ \rho^2 x_i^4 - (4\rho^2 + \delta\rho) x_i^2 + \rho^2 + \delta\rho \} \right. \\ + & \left. 2 \sum_{1 \le i < j \le p} (y_i - 1)(y_j - 1) \rho^2 H_2(x_i) H_2(x_j) \right] \varphi_p(\boldsymbol{x}). \end{aligned} \tag{3.25}$$

Since $H_4(x) = x^4 - 6x^2 + 3$, we get from (3.25) that

$$\begin{aligned} M - & \rho^2 \left(\partial'_{\boldsymbol{x}} (Y - I) \partial_{\boldsymbol{x}} \right)^2 \varphi_p(\boldsymbol{x}) \\ = & \sum_{i=1}^{p} (y_i - 1)^2 \left[(2\rho^2 - \delta\rho) x_i^2 + \delta\rho - 2\rho^2 \right] \varphi_p(\boldsymbol{x}) \\ = & \sum_{i=1}^{p} (y_i - 1)^2 (2\rho^2 - \delta\rho)(x_i^2 - 1) \varphi_p(\boldsymbol{x}) \\ = & (2\rho^2 - \delta\rho) \left(\partial'_{\boldsymbol{x}} (Y - I)^2 \partial_{\boldsymbol{x}} \right) \varphi_p(\boldsymbol{x}). \end{aligned}$$

Hence (3.23) is proved.

4. Asymptotic Expansions When A Is Symmetric

In this section we show how asymptotic expansions for $\mathrm{P}(\boldsymbol{X} \in A)$ given in Theorems 3.1∼ 3.3 can be written in the case when A is a symmetric set, that is, $A \subset \mathbf{R}^p$ stays the same for any permutation of cordintes $x_1 \ldots, x_p$. Put

$$\hat{\mathrm{P}}_{\delta,k,p}(A) = \int_A g_{\delta,k}(\boldsymbol{x})d\boldsymbol{x} = \sum_{j=0}^{k-1} \frac{1}{j!} M_j(A), \tag{4.1}$$

where $M_0(A) = \int_A g_p(\boldsymbol{x})d\boldsymbol{x}$ and for $j \geq 1$

$$\begin{aligned} M_j(A) &= \sum_{(j)} \frac{j!}{j_1! \ldots j_p!} \int_A b_{\delta,j_1}(x_1) \ldots b_{\delta,j_p}(x_p) g_p(\boldsymbol{x})d\boldsymbol{x} \\ &\quad \times \mathrm{E}\left[(Y_1 - 1)^{j_1} \ldots (Y_p - 1)^{j_p}\right]. \end{aligned} \tag{4.2}$$

Assume that A is symmetric with respect to $x_1, \ldots, x_p$ and let

$$\begin{aligned} I_{\delta,i} &= \int_A b_{\delta,i}(x_1) g_p(\boldsymbol{x})d\boldsymbol{x}, \\ I_{\delta,ij} &= \int_A b_{\delta,i}(x_1) b_{\delta,j}(x_2) g_p(\boldsymbol{x})d\boldsymbol{x}, \\ I_{\delta,ijk} &= \int_A b_{\delta,i}(x_1) b_{\delta,j}(x_2) b_{\delta,j}(x_3) g_p(\boldsymbol{x})d\boldsymbol{x}, \text{so on.} \end{aligned} \tag{4.3}$$

Then we have

$$\begin{aligned} M_1(A) &= I_{\delta,1}\mathrm{E}[\sum_{i=1}^p (Y_i - 1)], \\ M_2(A) &= I_{\delta,2}\mathrm{E}[\sum_{i=1}^p (Y_i - 1)^2] + I_{\delta,11}\mathrm{E}[\sum_{i \neq j}^p (Y_i - 1)(Y_j - 1)], \\ M_3(A) &= I_{\delta,3}\mathrm{E}[\sum_{i=1}^p (Y_i - 1)^3] + 3I_{\delta,21}\mathrm{E}[\sum_{i \neq j}^p (Y_i - 1)^2(Y_j - 1)] \\ &+ I_{\delta,111}\mathrm{E}[\sum_{i \neq j \neq k}^p (Y_i - 1)(Y_j - 1)(Y_k - 1)]. \end{aligned} \tag{4.4}$$

Assume that $Y_1 \geq \ldots \geq Y_p$ are the characteristic roots of a random matrix W. Let $V = W - I$. Then it is easily seen that

$$\sum_{i=1}^p (Y_i - 1)^j = \mathrm{tr}V^j, \quad j = 1, 2, \ldots,$$

$$
\begin{aligned}
&\sum_{i\neq j}^{p}(Y_i-1)(Y_j-1)=(\mathrm{tr}V)^2-\mathrm{tr}V^2,\\
&\sum_{i\neq j}^{p}(Y_i-1)^2(Y_j-1)=\mathrm{tr}V\mathrm{tr}V^2-\mathrm{tr}V^3, \qquad (4.5)\\
&\sum_{i\neq j\neq k}^{p}(Y_i-1)(Y_j-1)(Y_k-1)=(\mathrm{tr}V)^3+3\mathrm{tr}V\mathrm{tr}V^2-4\mathrm{tr}V^3.
\end{aligned}
$$

Note that the quantity $\sum_{i=1}^{p}|Y_i-1|^k$ appeared in our error bounds can be written as

$$
\begin{aligned}
\sum_{i=1}^{p}|Y_i-1|^k &= \sum_{i=1}^{p}(Y_i-1)^k\\
&= \mathrm{tr}V^k, \qquad (4.6)
\end{aligned}
$$

if k is even. If k is odd and $\mathrm{E}(Y_i^{k+1})<\infty$ for all $i=1,2,\ldots,p$, then using Holder's inequality we get

$$
\begin{aligned}
\sum_{i=1}^{p}|Y_i-1|^k &\le p^{1/(k+1)}\left(\sum_{i=1}^{p}(Y_i-1)^{k+1}\right)^{k/(k+1)}\\
&= p^{1/(k+1)}\left(\mathrm{tr}V^{(k+1)}\right)^{k/(k+1)}. \qquad (4.7)
\end{aligned}
$$

5. Hotelling's T_0^2-Statistic

In this section we consider error bounds for asymptotic approximatins of Lawley-Hotelling's T_0^2-statistic

$$
T_0^2=n\mathrm{tr}S_hS_e^{-1}, \qquad (5.1)
$$

where S_h and S_e are independently distributed as Wishart distributions $W_p(q,I_p)$ and $W_p(n,I_p)$, respectively. The statistic is used as one of the test statistics in multivariate linear model. The limiting of T_0^2 is a chi-square distribution with $r=pq$ degrees of freedom. Further, it is known (see, e.g., Anderson (1984)) that T_0^2 has an asymptotic expansion

$$
\begin{aligned}
P(T_0^2\le x) &= G_r(x)+\frac{r}{4n}\{(q-p-1)G_r(x)\\
&- 2qG_{r+2}(x)+(q+p+1)G_{r+4}(x)\}+\mathrm{O}(n^{-2}),
\end{aligned}
$$

where G_r is the distribution function of the chi-squared variate with r degrees of freedom.

Lemma 5.1. *We can write T_0^2 in terms of a multivariate scale mixture $\boldsymbol{X} = (X_1, \ldots, X_p)' = \mathrm{diag}(S_1, \ldots, S_p)\boldsymbol{Z}$ as*

$$T_0^2 = X_1 + \ldots + X_p, \tag{5.2}$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_p), Z_1, \ldots, Z_p$ are i.i.d. random variables, $Z_1 \sim \chi_q^2$, and letting $S_i = Y_i^{-1}(i = 1, \ldots, p)$, $Y_1 > \ldots > Y_p > 0$ are the characteristic roots of W such that $nW \sim W_p(n, I_p)$.

Proof. It is well known that the distribution T_0^2 can be expressed as

$$\begin{aligned} T_0^2 &= n\mathrm{tr}(U'U)S_e^{-1} \\ &= n\mathrm{tr}(H'U'UH)(H'S_eH)^{-1}, \end{aligned}$$

where U is a $q \times p$ random matrix whose elements are independent idntically distributed as $N(0, 1)$, and H is an orthogonal matrix. Note that the distributions of UH and $H'S_eH$ are the same as U and S_e, respectively. The result is obtained by choosing H such that $H'S_eH = \mathrm{diag}(Y_1, \ldots, Y_p)$.

Lemma 5.2. *Let $\boldsymbol{U} = (U_{11}, \ldots, U_{1q}, \ldots, U_{p1}, \ldots, U_{pq})$ be an $r = pq$ dimensional random vector such that*

$$\boldsymbol{U} = (\mathrm{diag}(S_1, \ldots, S_p) \otimes I_p)\boldsymbol{Z},$$

where $\boldsymbol{Z} = (Z_{11}, \ldots, Z_{1q}, \ldots, Z_{p1}, \ldots, Z_{pq}), Z_{ij}'s$ are i.i.d. random variables, $Z_{11} \sim N(0, 1)$, and letting $S_i = Y_i^{-1/2}(i = 1, \ldots, p)$, $Y_1 > \ldots > Y_p > 0$ are the characteristic roots of W such that $nW \sim W_p(n, I_p)$. Then we can also write T_0^2 as

$$T_0^2 = \boldsymbol{U}'\boldsymbol{U}. \tag{5.3}$$

Proof. This is a direct consequence of Lemma 5.1.

Now we show what kind of approximations and error bounds we can give for the distribution function of T_0^2 using Theorems 3.2 and 3.3. First we use the representation (5.2). In this case we apply Theorems 3.2 and 3.3 with $\delta = -1$ and $\rho = 1$.

Let $g_q(x)$ be a density function of χ_q^2, i.e.

$$g_q(x) = \frac{1}{2^{q/2}\Gamma(q/2)} x^{q/2-1} \exp(-x/2).$$

Then the functions $b_{-1,j}(x)$ defined by (3.3) are given by

$$\begin{aligned}
b_{-1,1}(x) &= -\frac{1}{2}(x-q), \\
b_{-1,2}(x) &= \frac{1}{4}\{x^2 - 2qx + q(q-2)\}, \\
b_{-1,3}(x) &= -\frac{1}{8}\{x^3 - 3qx^2 + 3q(q-2)x - (q-2)^2(q-4)\}, \\
b_{-1,4}(x) &= \frac{1}{16}\{x^4 - 4qx^3 + 6q(q-2)x^2 - 4q(q-2)(q-4)x \\
&\quad +q(q-2)(q-4)(q-6)\}.
\end{aligned} \tag{5.4}$$

It is easy to compute that

$$\begin{aligned}
b_{-1,1}(x)g_q(x) &= \frac{1}{2}q\left\{g_q(x) - g_{q+2}(x)\right\}, \\
b_{-1,2}(x)g_q(x) &= \frac{1}{4}q\left\{(q-2)g_q(x) - 2qg_{q+2}(x) + (q+2)g_{q+4}(x)\right\}, \\
b_{-1,3}(x)g_q(x) &= \frac{1}{8}q\left\{(q-2)(q-4)g_q(x) - 3q(q-2)g_{q+2}(x)\right. \\
&+ \left. 3q(q+2)g_{q+4}(x) - (q+2)(q+4)g_{q+6}(x)\right\}, \\
b_{-1,4}(x)g_q(x) &= \frac{1}{16}q\left\{(q-2)(q-4)(q-6)g_q(x)\right. \\
&- 4q(q-2)(q-4)g_{q+2}(x) + 6q(q+2)(q+4)g_{q+4}(x) \\
&- 8q(q+2)(q+4)g_{q+6}(x) \\
&+ \left.(q+2)(q+4)(q+6)g_{q+8}(x)\right\}.
\end{aligned}$$

Since

$$\mathrm{P}(T_0^2 \le x) = \mathrm{P}(\boldsymbol{X} \in A) \tag{5.5}$$

with $A = \{(x_1, \ldots, x_p) \in \mathbf{R}^p : \ x_1 + \ldots + x_p \le x\}$, the set A is symmetric with respect to $x_1, \ldots, x_p$. Therefore, applying the results in Section 4, we get

$$\hat{\mathrm{P}}_{-1,4,p}(A) = \sum_{j=0}^{3} \frac{1}{j!} M_j(A), \tag{5.6}$$

where

$$M_0(A) = \mathrm{P}(\boldsymbol{Z} \in A) = \mathrm{P}(\chi_r^2 \le x) = G_r(x)$$

with $r = pq$, and in the expressions for $M_i(A), i = 1, 2, 3$ in (4.4) we have to take

$$
\begin{aligned}
I_{-1,1} &= \frac{1}{2} q \left[G_r(x) - G_{r+2}(x) \right], \\
I{-1,2} &= \frac{1}{4} q \left\{ (q-2) G_r(x) - 2q G_{r+2}(x) + (q+2) G_{r+4}(x) \right\}, \\
I_{-1,11} &= \frac{1}{4} q^2 \left\{ G_r(x) - 2 G_{r+2}(x) + G_{r+4}(x) \right\}, \\
I_{-1,3} &= \frac{1}{8} q \left\{ (q-2)(q-4) G_r(x) - 3q(q-2) G_{r+2}(x) \right. \\
&\quad \left. + 3q(q+2) G_{r+4}(x) - (q+2)(q+4) G_{r+6}(x) \right\}, \\
I_{-1,21} &= \frac{1}{8} q^2 \left\{ (q-2) G_r(x) - 3(q-2) G_{r+2}(x) \right. \\
&\quad \left. + 3(q+2) G_{r+4}(x) - (q+2) G_{r+6}(x) \right\}, \\
I_{-1,111} &= \frac{1}{8} q^3 \left\{ G_r(x) - 3 G_{r+2}(x) + 3 G_{r+4}(x) - G_{r+6}(x) \right\}.
\end{aligned}
$$

Let $V = W - I$, where the characteristic roots of W are $Y_1, \ldots, Y_p$. Then we can use the following results:

$$
\begin{aligned}
&\mathrm{E}[\mathrm{tr} V] = 0, \quad \mathrm{E}[\mathrm{tr} V^2] = \frac{1}{n} p(p+1), \\
&\mathrm{E}[(\mathrm{tr} V)^2] = \frac{1}{n} 2p, \quad \mathrm{E}[\mathrm{tr} V^3] = \frac{1}{n^2} p(p^2 + 3p + 4), \\
&\mathrm{E}[\mathrm{tr} V \mathrm{tr} V^2] = \frac{4}{n^2} p(p+1), \quad \mathrm{E}[(\mathrm{tr} V)^3] = \frac{8}{n^2} p.
\end{aligned}
$$

Using the above results we obtain

$$
\begin{aligned}
\hat{\mathrm{P}}_{-1,2,p}(A) &= G_r(x), \\
\hat{\mathrm{P}}_{-1,2,p}(A) &= G_r(x) + \frac{r}{4n} \{ (q - p - 1) G_r(x) \\
&- 2q G_{r+2}(x) + (q + p + 1) G_{r+4}(x) \} \\
&+ \frac{r}{96 n^2} \{ () G_r(x) + () G_{r+2}(x) \\
&+ () G_{r+4}(x) + () G_{r+6}(x) \}.
\end{aligned}
$$

It follows from Theorems 3.2 and 3.3 that their error bounds are given as follows:

$$
|\mathrm{P}(T_0^2 \le x) - \hat{\mathrm{P}}_{-1,2,p}(A)|
$$

$$\leq \frac{1}{2n} p(p+1) \min\{\eta_{-1,2,p}, \nu_{-1,2,p}\}, \tag{5.7}$$

$$|\mathrm{P}(T_0^2 \leq x) - \hat{\mathrm{P}}_{-1,4,p}(A)|$$
$$\leq \frac{1}{2n^2} p(2p^2 + 5p + 5) \min\{\eta_{-1,4,p}, \nu_{-1,4,p}\}. \tag{5.8}$$

Next we consider asymptotic expansions and their error bounds based on the expression (5.3), i.e.

$$\mathrm{P}(T_0^2 \leq x) = \mathrm{P}(\boldsymbol{U} \in \tilde{A}), \tag{5.9}$$

where

$$\tilde{A} = \{(u_{11}, \ldots, u_{1q}, \ldots, u_{p1}, \ldots, u_{pq}) \in \mathbf{R}^r : \sum_{i,j} u_{ij}^2 \leq x\}.$$

In this case we use Theorems 3.2 and 3.3 with $\delta = -1$ and $\rho = 1/2$. However, it is necessary to make a slight modification, since the number of independent elements in the scale matrix is not r, but p. In fact, the term $\prod_{i=1}^p y_i^{-\delta\rho} g(x_i y_i^{-\delta\rho})$ in the M_j of (4.1) or (5.5) should be changed to

$$\prod_{i=1}^p \prod_{j=1}^q y_i^{1/2} \varphi(u_{ij} y_i^{1/2}) = \prod_{i=1}^p y_i^{q/2} (2\pi)^{-q/2} \exp(-\frac{1}{2} y_i u_i^2), \tag{5.10}$$

where $u_i^2 = \sum_{j=1}^q u_{ij}^2$. Let $\tilde{b}_{-1,j}$ be the function corresponding to the function $b_{\delta,j}$ in (4.2). Then, the function $\tilde{b}_{-1,j}$ is defined by

$$\frac{d^j}{dy^j} \left(y^{q/2} (2\pi)^{-q/2} \exp(-\frac{1}{2} y u^2) \right)$$
$$= y^{-j} \tilde{b}_{-1,j}(u^2 y^{1/2}) y^{q/2} (2\pi)^{-q/2} \exp(-\frac{1}{2} y u^2).$$

We denote the corresponding approximation by

$$\tilde{\mathrm{P}}_{-1,4,p}(\tilde{A}) = \sum_{j=0}^3 \frac{1}{j!} \tilde{M}_j(\tilde{A}). \tag{5.11}$$

It is easily seen that $b_{-1,j} = \tilde{b}_{-1,j}$. This result implies that the two approximations $\hat{\mathrm{P}}_{-1,4,p}(A)$ and $\tilde{\mathrm{P}}_{-1,4,p}(\tilde{A})$ are the same. Further, the corresponding error bounds are also the same, since

$$\mathrm{E}[|b_{-1,j}(X)|] = \mathrm{E}[|\tilde{b}_{-1,j}(U^2)|],$$

where X is distributed as χ^q, $U^2 = U_1^2 + \ldots + U_p^2$ and $U_1, \ldots, U_p$ are independent identically distributed as $N(0,1)$.

6. Lemmas

First we consider a basic result in the univariate case, i.e., $X = SZ = Y^{\delta\rho}Z$.

Lemma 6.1. *We have for any* $k \geq 1$

$$|G(x) - G_{\delta,k}(x)| \leq \beta_{\delta,k}\mathrm{E}\left[|Y-1|^k\right], \tag{6.1}$$

$$\|f(x) - g_{\delta,k}(x)\|_1 \leq \eta_{\delta,k}\mathrm{E}\left[|Y-1|^k\right], \tag{6.2}$$

where $\beta_{\delta,k} = \beta_{\delta,k,1}$.

Proof. For a proof of (6.1), see Shimizu and Fujikoshi (1997). In the following we prove (6.2). Here and in following we use an expansion formula for a function h with $k \geq 1$ continuous derivatives

$$\begin{aligned} h(y) = h(1) &+ \sum_{j=1}^{k-1} \frac{1}{j!} h^{(j)}(1)(y-1)^j \\ &+ \frac{(y-1)^k}{(k-1)!}\mathrm{E}\left[(1-\tau)^{k-1}h^{(k)}(1+\tau(y-1))\right], \end{aligned} \tag{6.3}$$

where τ is an uniformly distributed $[0,1]$ random variable.

For any $y > 0$ let

$$\Delta_{\delta,k}(x,y) \equiv y^{-\delta\rho}g(xy^{-\delta\rho}) - g_{\delta,k}(x,y). \tag{6.4}$$

Using (3.3), (6.3) and (6.4) we can write also for $k \geq 1$

$$\begin{aligned} \Delta_{\delta,k}(x,y) = &\frac{(y-1)^k}{(k-1)!}\mathrm{E}\left[(1-\tau)^{k-1}\left(1+\tau(y-1)\right)^{-k-\delta\rho}\right. \\ &\left.\times b_{\delta,k}\left(x(1+\tau(y-1))^{-\delta\rho}\right) g\left(x(1+\tau(y-1))^{-\delta\rho}\right)\right]. \end{aligned} \tag{6.5}$$

The idea of our proof is to use (6.4) or (6.5) depending on whether y is far from 1 or close to it. Let

$$\varphi = (\xi_{\delta,k}/\eta_{\delta,k})^{1/k}.$$

Note that $\varphi : 0 < \varphi < 1$. If $y : 0 < y < \varphi$, then it follows from (6.3) that

$$\begin{aligned} \|\Delta_{\delta,k}(x,y)\|_1 &\leq \left(1 + \sum_{j=0}^{k-1}\xi_{\delta,j}\right)\frac{(1-y)^k}{(1-\varphi)^k} \\ &= \eta_{\delta,k}|y-1|^k. \end{aligned} \tag{6.6}$$

If $y \geq \varphi$, then for any $\tau \in [0,1]$ we have $1+\tau(y-1) \geq \varphi$. Therefore it follows from (6.5) and Fubini theorem that

$$\|\Delta_{\delta,k}(x,y)\|_1 \leq \xi_{\delta,k}\frac{|y-1|^k}{\varphi^k} = \eta_{\delta,k}|y-1|^k. \tag{6.7}$$

Combining (6.4), (6.6) and (6.7) we get (6.2).

Lemma 6.2. *Let $0 \leq j_1 \leq j_2 \leq \cdots \leq j_p$ be integers such that $j_1 + \cdots + j_p = j$ and $a_1, \cdots, a_p$ be non-negative real numbers. Then*

$$\sum a_1^{i_1} \cdots a_p^{i_p} \leq (p-1)!(a_1^j + \cdots + a_p^j), \tag{6.8}$$

where summation on the left-hand side is taken over all $p!$ permutations $(i_1, \cdots, i_p)$ of $(j_1, \cdots, j_p)$.

Remark 6.1. Note when $a_1 = \cdots = a_p = 1$ inequality (6.8) is written in the form $p! \leq p!$. Therefore, (6.8) is sharp.

Proof. We prove (6.8) by mathematical induction on p. If $p=1$, then (6.8) is obvious. We assume that (6.8) is valid for $p-1 \geq 1$. We write the left-hand side of (6.8) in the form

$$\sum a_1^{i_1} \ldots a_p^{i_p} = a_1^{j_1} P_{j-j_1}(a_2, \cdots, a_p) + \ldots + a_1^{j_p} P_{j-j_p}(a_2, \ldots, a_p),$$

where $P_{j-j_1}(a_2, \ldots, a_p) = \sum a_2^{i_2} \ldots a_p^{i_p}$ and the summation here is taken over all $(p-1)!$ permutations $(i_2, \cdots, i_p)$ of $(j_2, \ldots, j_p)$. Polynomials $P_{j-j_2}, \ldots, P_{j-j_p}$ are defined similarly. The hypothesis of the induction asserts that for all $\ell = 1, 2, \ldots, p$ we have

$$P_{j-j_\ell}(a_2, \ldots, a_p) \leq (p-2)!(a_2^{j-j_\ell} + \ldots + a_p^{j-j_\ell}).$$

Therefore we get

$$\begin{aligned}\sum a_1^{i_1} \ldots a_p^{i_p} \leq (p-2)! \Big[a_1^{j_1}\left(a_2^{j-j_1} + \ldots + a_p^{j-j_1}\right) + \cdots \\ + a_1^{j_p}\left(a_2^{j-j_p} + \ldots + a_p^{j-j_p}\right)\Big].\end{aligned} \tag{6.9}$$

It is clear that on the right-hand side of (6.9) we can replace a_1 by any other a_i, that is, for $i = 1, 2, \cdots, p$ we have

$$\sum a_1^{i_1} \cdots a_p^{i_p} \leq (p-2)! \left[a_i^{j_1} \sum_{n=1, n\neq i}^{p} a_n^{j-j_1} + \cdots + a_i^{j_p} \sum_{n=1, n\neq i}^{p} a_n^{j-j_p} \right]. \tag{6.10}$$

Note that for any positive b_1 and b_2 a function $b_1^{j-x}b_2^x + b_1^x b_2^{j-x}$ of x is convex on $[0, j]$ and is equal to $b_1^j + b_2^j$ for $x = 0$ and $x = j$ (cf. Lemma 2 in Shimizu (1995)). Therefore for all integers $i = 0, 1, \cdots, j$ we have

$$b_1^{j-i}b_2^i + b_1^i b_2^{j-i} \le b_1^j + b_2^j. \tag{6.11}$$

Thus summing up inequalities (6.10) for $i = 1, 2, \cdots, p$ we get from (6.11) that

$$p \sum a_1^{i_1} \cdots a_p^{i_p} \le (p-2)!(p-1)p(a_1^k + \cdots + a_p^k). \tag{6.12}$$

Hence we obtain (6.8).

Lemma 6.3. *Assume that $G(x)$ satisfies* A1. *Let $j (0 < j \le k)$ be a positive integer and*

$$M(\boldsymbol{x}) = \sum_{(j)} j! \prod_{i=1}^{p} a_i^{j_i} \frac{1}{j_i!} y_{ij_i}^{-j_i} c_{\delta,j_i}(x_i y_{ij_i}^{-\delta\rho}) g(x_i y_{ij_i}^{-\delta\rho}),$$

where $a_1, \cdots, a_p$ are positive numbers and $\sum_{(j)}$ means summation over all non-negative integers $j_1, \ldots, j_p$ such that $j_1 + \ldots + j_p = j$. If all $y_{ij_i} \ge \varphi > 0$, $i = 1, 2, \ldots, p$, then

$$|M(x)| \le (a_1^j + \ldots + a_p^j)\varphi^{-j} W_{\delta,j,p} \tag{6.13}$$

$W_{\delta,j,p}$ is defined by (2.3).

Proof. Since all $y_{ij_i} \ge \varphi, i = 1, 2, \ldots, p$, and for any permutation of $(j_1, \ldots, j_p)$ a product $\alpha_{\delta,j_1} \ldots \alpha_{\delta,j_p}$ does not change, we get

$$|M(\boldsymbol{x})| \le \varphi^{-j} \sum_{[j]} \left(\alpha_{j_1} \ldots \alpha_{j_p} \sum_1 a_1^{\ell_1} \ldots a_p^{\ell_p} \right), \tag{6.14}$$

where $\sum_{[j]}$ denotes summation over all non-negative integers $0 \le j_1 \le \ldots \le j_p$ such that $j_1 + \ldots + j_p = j$, and $\sum_1$ means summation over all different permutations $\{\ell_1, \ell_2, \cdots, \ell_p\}$ of a fixed set $\{j_1, j_2, \cdots, j_p\}$. Since $\sum_1$ consists of $p!/(i_1! \ldots i_m!)$ summands, where $i_1, \ldots, i_m$ are such that m and $i_1, \ldots, i_m$ are positive integers satisfing

$$\begin{aligned} &0 \le j_1 = \ldots = j_{i_1} < j_{i_1+1} = \ldots = j_{i_1+i_2} \\ &< \ldots < j_{i_1+\ldots+i_{m-1}+1} = \ldots = j_{i_1+\ldots+i_m} (= j_p) \le j. \end{aligned}$$

we get (6.13) from (6.14) and Lemma 6.2.

Lemma 6.4. *Let*

$$H(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}).$$

Assume that $G(x)$ *satisfies* A1. *Then for any positive* $a_1,\ldots,a_p$ *and a positive integer* $j : 0 < j \le k$ *we have*

$$\left|\left(a_1\frac{\partial}{\partial y_1}+\ldots+a_p\frac{\partial}{\partial y_p}\right)^j H(\boldsymbol{x},\boldsymbol{y})\bigg|_{y_i=y_{i0},i=1,2,\ldots,p}\right| \le (a_1^j+\ldots+a_p^j)\frac{j!}{\varphi^j}W_{\delta,j,p}, \tag{6.15}$$

provided $y_{i0} \ge \varphi > 0$, $i = 1,2,\ldots,p$, *where* $W_{\delta,j,p}$ *is defined by* (2.3).

Proof. It follows immediately from Lemma 6.3 and the fact that the left-hand side equals

$$\sum_{(j)} j! \prod_{i=1}^{p} a_i^{j_i}\frac{1}{j_i!}y_{i0}^{-j_i}c_{\delta,j_i}(x_i y_{i0}^{-\delta\rho})g(x_i y_{i0}^{-\delta\rho}).$$

For any integrable function $F(\boldsymbol{x}) : \mathbf{R}^p \to \mathbf{R}^1$, we define

$$\|F(\boldsymbol{x})\|_p = \int_{\mathbf{R}^p}|F(\boldsymbol{x})|d\boldsymbol{x}.$$

Lemma 6.5. *Assume that* $g(x)$ *satisfies* A2. *Let* $j(0 < j \le k)$ *be a positive integer and*

$$I(\boldsymbol{x}) = \sum_{(j)} j! \prod_{i=1}^{p} a_i^{j_i}\frac{1}{j_i!}y_{ij_i}^{-j_i}b_{\delta,j_i}(x_i y_{ij_i}^{-\delta\rho})g(x_i y_{ij_i}^{-\delta\rho}),$$

where $a_1,\cdots,a_p$ *are positive numbers. If all* $y_{ij_i} \ge \varphi > 0$, $i = 1,2,\ldots,p$, *then*

$$\|I(x)\|_p \le (a_1^j+\ldots+a_p^j)\varphi^{-j}V_{\delta,j,p} \tag{6.16}$$

$V_{\delta,j,p}$ *is defined by* (3.10).

Proof. The arguments are the same as in Lemma 6.3. Since all $y_{ij_i} \geq \varphi$ and for any permutation of $(j_1, \ldots, j_p)$ a product $\xi_{\delta,j_1} \ldots \xi_{\delta,j_p}$ does not change, we get

$$\|I(\boldsymbol{x})\|_p \leq \varphi^{-j} \sum_{[j]} \left(\xi_{j_1} \ldots \xi_{j_p} \sum_{1} a_1^{\ell_1} \ldots a_p^{\ell_p} \right), \tag{6.17}$$

Thus, Lemma 6.5 and (6.16) impl (6.15).

Lemma 6.6. *Let*

$$h(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{p} y_i^{-\delta\rho} g(x_i y_i^{-\delta\rho}).$$

Assume that $g(x)$ satisfies A2. *Then for any positive $a_1, \ldots, a_p$ and a positive integer $j : 0 < j \leq k$ we have*

$$\left\| \left(a_1 \frac{\partial}{\partial y_1} + \ldots + a_p \frac{\partial}{\partial y_p} \right)^j H(\boldsymbol{x}, \boldsymbol{y}) \Big|_{y_i = y_{i0}, i=1,2,\ldots,p} \right\| \leq (a_1^j + \ldots + a_p^j) \frac{j!}{\varphi^j} V_{\delta,j,p}, \tag{6.18}$$

provided $y_{i0} \geq \varphi > 0$, $i = 1, 2, \ldots, p$, where $V_{\delta,j,p}$ is defined by (3.10).

Proof. It follows immediately from Lemma 6.5 and the fact that the left-hand side equals

$$\sum_{(j)} j! \prod_{i=1}^{p} a_i^{j_i} \frac{1}{j_i!} y_{i0}^{-j_i} b_{\delta,j_i}(x_i y_{i0}^{-\delta\rho}) g(x_i y_{i0}^{-\delta\rho}).$$

7. Proof of Theorems

Proof of Theorems 2.1. Note that

$$F_p(\boldsymbol{x}) = \mathrm{E}\left[H(\boldsymbol{x}, \boldsymbol{Y})\right],$$

where $\boldsymbol{Y} = (Y_1, \ldots, Y_p)$ and a function H is defined in (6.13). We use a Taylor formula for a function H with $k \geq 1$ continuous derivatives

$$H(y) = H(1) + \sum_{j=1}^{k-1} \frac{1}{j!} H^{(j)}(1)(y-1)^j + \frac{1}{k!} H^{(k)}(1 + \tau(y-1))(y-1)^k, \tag{7.1}$$

where τ is a number on $(0,1)$. We construct an expansion for H using (7.1) sequentially. Namely, at first we apply (7.1) to $G(x_1 y_1^{-\delta\rho})$. We get

$$\begin{aligned} H(\boldsymbol{x},\boldsymbol{y}) &= \left[G(x_1) + \sum_{j=1}^{k-1} \frac{1}{j!} c_{\delta,j}(x_1) g(x_1)(y_1-1)^j + R_1 (y_1-1)^k \right] \\ &\times \ H_2(\boldsymbol{x},\boldsymbol{y}), \end{aligned} \tag{7.2}$$

where

$$R_1 = \frac{1}{k!} \frac{\partial}{\partial y^k} \left(G(x_1 y^{-\delta\rho}) \right) \Big|_{y=1+\tau(y_1-1)}$$

and

$$H_2(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=2}^{p} G(x_i y_i^{-\delta\rho}).$$

Now we apply (7.1) for a function $G(x_2 y_2^{-\delta\rho})$ so that for a summand

$$\frac{1}{j!} c_{\delta,j}(x_1) g(x_1)(y_1-1)^j H_2(\boldsymbol{x},\boldsymbol{y}),$$

we apply (7.1) with k replaced by $k-j$. At last we obtain the following expansion

$$h(\boldsymbol{x},\boldsymbol{y}) = g(x_1)\dots g(x_p) + \sum_{j=1}^{k-1} \sum_{(j)} \prod_{i=1}^{p} \frac{1}{j_i!} c_{\delta,j_i}(x_i) g(x_i)(y_1-1)^j + R_{\delta,k,p}, \tag{7.3}$$

where $R_{\delta,k,p}$ is a sum of terms each of which can be written in the form

$$(y_1-1)^{k_1} \cdots (y_p-1)^{k_p} M_{k_1}(y_1) \dots M_{k_p}(y_p) \tag{7.4}$$

with $k_i \geq 0$ for $i = 1, 2, \cdots, p$ and $k_1 + \ldots + k_p = k$. Each factor M_j in (7.4) has one of the following form:

$$M_k(y) = \frac{1}{k!} \frac{\partial^k}{\partial y_1^k} \left(G(x y_1^{-\delta\rho}) \right) \Big|_{y_1=1+\tau(y-1)}, \tag{7.5}$$

$M_0(y) = G(x)$ or $M_0(y) = G(xy^{-\delta\rho})$ and when $j : 1 \leq j \leq k-1$, we have for $M_j(y)$ one of the two representations:

$$\frac{1}{j!} c_{\delta,j}(x) g(x) \quad \text{or} \quad \frac{1}{j!} \frac{\partial^j}{\partial y_1^j} \left(G(x y_1^{-\delta\rho}) \right) \Big|_{y_1=1+\tau(y-1)}. \tag{7.6}$$

Let

$$\varphi_1 = (W_{\delta,k,p}/\eta_{\delta,k,p})^{1/k}\,. \tag{7.7}$$

At first we consider the case when $0 < \min(y_1, \ldots, y_p) \le \varphi_1$. Assume that y_1 is such that $0 < y_1 \le \varphi_1$. We have for any $j : 1 \le j \le k$,

$$\begin{aligned}
&|1-y_1|^j + \ldots + |1-y_p|^j \\
&\le \frac{1}{(1-\varphi_1)^{k-j}} \Big(|1-y_1|^k \\
&+ |1-y_1|^{k-j}|1-y_2|^j + \ldots + |1-y_1|^{k-j}|1-y_p|^j \Big) \\
&\le \frac{p}{(1-\varphi_1)^{k-j}} \Big(|1-y_1|^k + \ldots + |1-y_p|^k \Big)
\end{aligned} \tag{7.8}$$

Therefore, using Lemma 6.3, (7.3) and (7.7) we get

$$\begin{aligned}
|R_{\delta,k,p}| &\le 2 + \sum_{j=1}^{k-1} \Big(|1-y_1|^j + \ldots + |1-y_p|^j \Big) W_{\delta,j,p} \\
&\le \frac{1}{(1-\varphi_1)^k} \Big(|1-y_1|^k + \ldots + |1-y_p|^k \Big) \\
&\qquad \times \left(2 + \sum_{j=1}^{k-1} W_{\delta,j,p} \right) \\
&= \eta_{\delta,k,p} \Big[|1-y_1|^k + \ldots + |1-y_p|^k \Big]\,.
\end{aligned} \tag{7.9}$$

If $\min(y_1, \cdots, y_p) > \varphi_1$ then using Lemma 6.4, (7.4) and representations for summands contained in $R_{\delta,k,p}$ we get

$$\begin{aligned}
|R_{\delta,k,p}| &\le \frac{W_{\delta,k,p}}{\varphi_1^k} \Big[|1-y_1|^k + \cdots + |1-y_p|^k \Big] \\
&= \eta_{\delta,k,p} \Big[|1-y_1|^k + \ldots + |1-y_p|^k \Big]\,.
\end{aligned} \tag{7.10}$$

According to remark in the beginning of the proof and combining (7.8) and (7.9) we finish the proof of Theorem 2.1.

Proof of Theorems 2.2. The result can be proved by using arguments similar to the proof of Lemma 2 in Shimizu (1995). In order to prove (2.6) it is enough as usual to show that

$$\left| \prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}) - G_{\delta,k,p}(x) \right| \le \gamma_{\delta,k,p} \sum_{i=1}^{p} |y_i - 1|^k, \tag{7.11}$$

where $G_{\delta,k,p}$ is defined by (2.2) but $y_i, i = 1, \ldots, p$, are considered as positive real numbers.

We prove (7.10) by mathematical induction with respect to p. In the case $p = 1$ the inequlity (7.10) was proved in Theorem 2.1 of Shimizu and Fujikoshi (1997). Therefore, we can write for $p \geq 2$

$$\prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}) = \left[G(x_p) + \sum_{j=1}^{k-1} \frac{1}{j!} (y_j - 1)^j c_{\delta,j}(x) g(x) + R_{\delta,p} \right] \times \prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho}), \tag{7.12}$$

where $|R_{\delta,p}| \leq \beta_{\delta,k} |y_p - 1|^k$. Assume that (7.10) holds for $p - 1$. Then we apply (7.10) to $\prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho})$ with p replaced by $p - 1$ and k replaced by $k - j$ when $\prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho})$ is a factor by $(y_p - 1)^j$ in (7.11). Thus, we get

$$\left| \prod_{i=1}^{p} G(x_i y_i^{-\delta\rho}) - G_{\delta,k,p}(x) \right| \leq \beta_{\delta,k} |y_p - 1|^k + \sum_{q=0}^{k-1} \alpha_{\delta,q} |y_p - 1|^q \gamma_{\delta,k-q,p-1} \sum_{i=1}^{p} |y_i - 1|^{k-q}. \tag{7.13}$$

We got (7.13) from (7.12) applying induction hypothesis to $\prod_{i=1}^{p-1} G(x_i y_i^{-\delta\rho})$. It is clear we could use the same arguments to the function $\prod_{i=1, i\neq j}^{p} G(x_i y_i^{-\delta\rho})$ with any $j = 1, 2, \ldots, p$. Then we could get (7.13) with $|y_p - 1|$ with $|y_p - 1|$ replaced by $|y_j - 1|$. Since in all these inequalities the left-hand sides will coincide, summing up the inequalities for $j = 1, 2, \ldots, p$ and applying (6.12) (cf. the proof of Lemma 2 in Shimizu (1995)) we come to (2.7) and recurrence formula for $\gamma_{\delta,k,p}$ stated in Theorem 2.2.

Proof of Theorem 3.1. This is a direct consequence of the inequality (6.3) in Lemma 6.1.

Proof of Theorem 3.2. The proof is similar to the one of Theorem 2.2. Note that

$$f_p(\boldsymbol{x}) = \mathrm{E}\left[h(\boldsymbol{x}, \boldsymbol{Y}) \right],$$

where $\boldsymbol{Y} = (Y_1, \ldots, Y_p)$ and a function h is defined in (6.16). We construct an expansion for h using (6.3) sequentially. Namely, at first we apply (6.3) to $y_1^{-\delta\rho} g(x_1 y_1^{-\delta\rho})$. We get

$$h(\boldsymbol{x},\boldsymbol{y}) = \left[g(x_1) + \sum_{j=1}^{k-1} \frac{1}{j!} b_{\delta,j}(x_1) g(x_1)(y_1-1)^j + R_1 (y_1-1)^k \right] \times h_2(\boldsymbol{x},\boldsymbol{y}), \tag{7.14}$$

where

$$R_1 = \frac{1}{(k-1)!} \mathrm{E} \left[(1-\tau)^{k-1} \frac{\partial}{\partial y^k} \left(y^{-\delta\rho} g(x_1 y^{-\delta\rho}) \right) \Big|_{y=1+\tau(y_1-1)} \right]$$

and

$$h_2(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=2}^{p} y_i^{-\delta\rho} g(x_i y_i^{-\delta\rho}).$$

Now we apply (6.3) for a function $y_2^{-\delta\rho} g(x_2 y_2^{-\delta\rho})$ so that for a summand

$$\frac{1}{j!} b_{\delta,j}(x_1) g(x_1)(y_1-1)^j h_2(\boldsymbol{x},\boldsymbol{y}),$$

we apply (6.3) with k replaced by $k-j$. At last we obtain the following expansion

$$h(\boldsymbol{x},\boldsymbol{y}) = g(x_1)\dots g(x_p) + \sum_{j=1}^{k-1} \sum_{(j)} \prod_{i=1}^{p} \frac{1}{j_i!} b_{\delta,j_i}(x_i) g(x_i)(y_1-1)^j + R_{\delta,k,p}, \tag{7.15}$$

where $R_{\delta,k,p}$ is a sum of terms each of which can be written in the form

$$(y_1-1)^{k_1} \cdots (y_p-1)^{k_p} I_{k_1}(y_1) \cdots I_{k_p}(y_p) \tag{7.16}$$

with $k_i \geq 0$ for $i = 1,2,\cdots,p$ and $k_1 + \cdots + k_p = k$. Each factor I_j in (7.15) has one of the following form:

$$I_k(y) = \frac{1}{(k-1)!} \int_0^1 (1-\tau)^{k-1} \frac{\partial^k}{\partial y_1^k} \left(y_1^{-\delta\rho} g(x y_1^{-\delta\rho}) \right) \Big|_{y_1=1+\tau(y-1)} d\tau, \tag{7.17}$$

$I_0(y) = g(x)$ or $I_0(y) = y^{-\delta\rho} g(x y^{-\delta\rho})$ and when $j : 1 \leq j \leq k-1$, we have for $I_j(y)$ one of the two representations:

$$\frac{1}{j!} b_{\delta,j}(x) g(x) \quad \text{or}$$

$$\frac{1}{(j-1)!} \int_0^1 (1-\tau)^{j-1} \frac{\partial^j}{\partial y_1^j} \left(y_1^{-\delta\rho} g(x y_1^{-\delta\rho}) \right) \Big|_{y_1=1+\tau(y-1)} d\tau.$$

Let

$$\varphi_1 = (V_{\delta,k,p}/\eta_{\delta,k,p})^{1/k}. \tag{7.18}$$

At first we consider the case when $0 < \min(y_1, \ldots, y_p) \leq \varphi_1$. Assume that y_1 is such that $0 < y_1 \leq \varphi_1$. Then, the same arguments as in (7.7) and (7.8) imply that for any $j : 1 \leq j \leq k$,

$$|1-y_1|^j + \ldots + |1-y_p|^j \leq \frac{p}{(1-\varphi_1)^{k-j}} \left(|1-y_1|^k + \ldots + |1-y_p|^k\right), \tag{7.19}$$

and

$$\|R_{\delta,k,p}\|_p \leq \eta_{\delta,k,p} \left[|1-y_1|^k + \ldots + |1-y_p|^k\right]. \tag{7.20}$$

Similarly, if $\min(y_1, \cdots, y_p) > \varphi_1$, then we have

$$\|R_{\delta,k,p}\|_p \leq \eta_{\delta,k,p} \left[|1-y_1|^k + \ldots + |1-y_p|^k\right]. \tag{7.21}$$

These results imply the consequence of Theorem 3.2.

Proof of Theorems 3.3. It is enough to repeat arguments of Lemma 2 and Theorem 2 in Shimizu (1995) replacing Lemma 1 in Shimizu (1995) by our Lemma 6.1.

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